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GCSE – Additional

Years 11 & 12

By R.M. O'Toole

B.A., M.C., M.S.A.

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The Author

R.M. O'Toole B.A., M.C., M.S.A.

The author is an experienced State Examiner, teacher, lecturer, consultant and author in mathematics.
She is also a member of The Society of Authors.

First Published in 1996 (Revised in 2006) by:

Mathematics Publishing Company,
'The Cottage',
45, Blackstaff Road,
Clough,
Downpatrick,
Co. Down,
BT30 8SR.

Tel. 028 44 851818 Fax. 028 44 851818

email address mathsbooks@hotmail.co.uk

ISBN No. 1900043 556

Testimonials

Below is a selection of written comments received from people who have used our material.

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Extracts from a head teacher's letter:

'… *very well received by parents, teachers and pupils …*'
'… *self contained…*'
'… *highly structured …*'
'… *all children including the less well able are helped …*'
'…*to develop concepts through a series of clearly defined steps …*'
'… *increased confidence for pupils …*'
'… *parents find user friendly as worked examples are given …*'
'… *language and notation are simple and clearly defined …*'

From a 10 year old pupil (boy):

'… *the material describes the working out in a way that is easy. The worked examples are laid out very clearly …*'

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From a 15 year old pupil (boy):

'… *simple and easy way to learn maths …*'
'… *careful explanations of each topic…*'
'… *also questions to make sure you know and understand what you have learned, and each question has a worked answer to check everything you have covered …*'
'… *so you are never left without any help …*'

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'…*easy to understand …*'
'…*clear and concise …*'
'…*thoroughly recommended …*'

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‘… *clearly explained and easy to understand* …’
‘… *well laid out* …’
‘… *well structured* …’
‘… *I would not hesitate to use these again* …’

From a 16 year old pupil (boy):

‘… *self-explanatory and easy* …’
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‘… *helped me consolidate* …’
‘… *succinct and effective* …’
‘… *boosted my confidence* …’
‘… *contributed significantly towards helping me to prepare for exams* …’

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‘… *your material… gave me help and reinforcement* …’
‘… *increasing my confidence to pursue my maths* …’
‘… *I am now enjoying life at university* …’

Acknowledgements

Special thanks to Cormac O'Toole, who advised on the computer technical end and to Denise Tumelty for design of covers.

Finally, I would like to thank all our customers for buying our books and for their kind letters of appreciation.

G.B. O'Toole, B.A. (Hons.), CertPFS

Publisher.

Mathematics Publishing Company.

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Dedication

I dedicate this work to the memory of my late parents, Jack and Mary Tumelty.

Ros.

GCSE (ADDITIONAL) – CONTENTS

(Exercise with worked answers included)

Section No. **Topic**

SECTION 1

MATRICES

A **matrix** is a rectangular array of numbers,
containing **m rows** and **n columns**.

The numbers in a matrix are called **elements** or **members**.

ORDER OF A MATRIX

Generally speaking, a matrix with **m rows** and **n columns**, is said to have **order m × n**.

E.g. $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is a matrix of **order 2 × 1**,

$(0\ 1)$ is a matrix of **order 1 × 2**,

$\begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$ is a matrix of **order 2 × 2**,

and so on.

Usually, a matrix is denoted by a **CAPITAL** letter:

E.g. $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$.

OPERATIONS ON MATRICES

The normal operations of **addition**, **subtraction**, **multiplication** and (the equivalent to) **division** may be performed on matrices, subject to certain conditions.

(i) Addition and Subtraction of Matrices

Matrices of the **same order** may be **added** or **subtracted**.
Simply **add** or **subtract** corresponding **elements**
to find the **sum** of, or the **difference** between, two matrices :

E.g. $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -2 & 1 \\ 0 & 3 \end{pmatrix}$

(a) $\mathbf{A} + \mathbf{B} = \begin{pmatrix} 2 + (-2) & -1 + 1 \\ 3 + 0 & 0 + 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 3 & 3 \end{pmatrix}$

(b) $\mathbf{A} - \mathbf{B} = \begin{pmatrix} 2 - (-2) & -1 - 1 \\ 3 - 0 & 0 - 3 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 3 & -3 \end{pmatrix}$

(ii) Multiplication By A Scalar

A matrix may be **multiplied** by **any scalar k**, i.e. a real number :

E.g. $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix}$

$$\mathbf{2A} = \begin{pmatrix} 1 \times 2 & -2 \times 2 \\ 3 \times 2 & 0 \times 2 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ 6 & 0 \end{pmatrix}$$

or $$\sqrt{2}\,\mathbf{A} = \begin{pmatrix} 1 \times \sqrt{2} & -2 \times \sqrt{2} \\ 3 \times \sqrt{2} & 0 \times \sqrt{2} \end{pmatrix} = \begin{pmatrix} \sqrt{2} & -2\sqrt{2} \\ 3\sqrt{2} & 0 \end{pmatrix}$$

and more generally,

$$\mathbf{kA} = \begin{pmatrix} k & -2k \\ 3k & 0 \end{pmatrix}.$$

(iii) Multiplication by another matrix

A matrix of **order m $\times$ n** may be **multiplied** by **another matrix** of **order n $\times$ p** and the **resultant** matrix is of **order m $\times$ p**.

This means that the **first matrix** must have the **same number** of **columns** as the **second matrix** has of **rows**, for matrix multiplication to be possible.

The **resultant matrix** has the **same number** of **rows** as the **first matrix** and the **same number** of **columns** as the **second matrix** :

E.g. $\mathbf{A} = \begin{pmatrix} \mathbf{2} & \mathbf{-1} \\ \mathbf{3} & \mathbf{1} \end{pmatrix}$, $\quad \mathbf{B} = \begin{pmatrix} \mathbf{2} & \mathbf{0} & \mathbf{3} \\ \mathbf{1} & \mathbf{-2} & \mathbf{-1} \end{pmatrix}$

A is of **order 2 $\times$ 2** and **B** is of **order 2 $\times$ 3**.

Therefore, **AB** is possible,

since **A** has **2 columns** and **B** has **2 rows**.

The **resultant matrix** has **2 rows** and **3 columns**,

but **BA** is **impossible**,

since **B** has **3 columns** and **A** has only **2 rows**.

To do **matrix multiplication**, each **row** of the **first** matrix is **multiplied** by each **column** of the **second** matrix :

THE MATRIX MULTIPLICATION FORMULA

Let $\mathbf{A} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} \mathbf{e} & \mathbf{f} \\ \mathbf{g} & \mathbf{h} \end{pmatrix}$

Then $\mathbf{AB} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}\begin{pmatrix} \mathbf{e} & \mathbf{f} \\ \mathbf{g} & \mathbf{h} \end{pmatrix} = \begin{pmatrix} \mathbf{ae+bg} & \mathbf{af+bh} \\ \mathbf{ce+dg} & \mathbf{cf+dh} \end{pmatrix}$

and $\mathbf{BA} = \begin{pmatrix} \mathbf{e} & \mathbf{f} \\ \mathbf{g} & \mathbf{h} \end{pmatrix}\begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} = \begin{pmatrix} \mathbf{ea+fc} & \mathbf{eb+fd} \\ \mathbf{ga+hc} & \mathbf{gb+hd} \end{pmatrix}$.

Applying the **matrix multiplication formula** to **A** and **B**,

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 & 0 & 3 \\ 1 & -2 & -1 \end{pmatrix}$$

we have :

$$\mathbf{AB} = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 3 \\ 1 & -2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2\times 2 + (-1)\times 1 & 2\times 0 + (-1)\times(-2) & 2\times 3 + (-1)\times(-1) \\ 3\times 2 + 1\times 1 & 3\times 0 + 1\times(-2) & 3\times 3 + 1\times(-1) \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 2 & 7 \\ 7 & -2 & 8 \end{pmatrix}$$

Since **BA** is **impossible**, it is clear that
matrices are **not commutative under multiplication**,

i.e. $\mathbf{AB} \neq \mathbf{BA}$.

(Here, **matrix** algebra **differs** from **ordinary** algebra, where
AB = BA.)

(iv) Matrix Division - Inverse Matrix

Multiplication by the **inverse matrix** is the **equivalent** of **division** in matrix algebra.

This compares with ordinary algebra where **division** by a quantity is the **same** as **multiplication** by the **reciprocal** of the quantity,

E.g. **Division** by x is **equivalent** to **multiplication** by $\frac{1}{x}$:

Compare **ordinary algebra**: $\quad xp = s \quad \Rightarrow \quad p = s \times \frac{1}{x} = \frac{s}{x}$

with **matrix algebra** : $\quad X\mathbf{B} = \mathbf{C} \quad \Rightarrow \quad \mathbf{B} = X^{-1}\mathbf{C}$,

where X^{-1} is the **inverse matrix** of **X**.

THE INVERSE MATRIX FORMULA

If $\mathbf{A} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}$

the **inverse matrix** of **A** is $\mathbf{A}^{-1}$

where $\mathbf{A}^{-1} = \dfrac{\mathbf{1}}{\mathbf{ad - bc}} \begin{pmatrix} \mathbf{d} & \mathbf{-b} \\ \mathbf{-c} & \mathbf{a} \end{pmatrix}$, **(ad-bc $\neq$ 0).**

The **ad-bc** in the **Inverse Matrix Formula** is called the **DETERMINANT** as it 'determines' whether a matrix can be inverted.
Since **division by zero** is **undefined**,
ad-bc cannot be equal to **zero**.

Singular and non-singular matrices

A matrix which **can** be inverted is called a **non-singular matrix** and a matrix which **cannot** be inverted is called a **singular matrix**.

If we take two matrices, $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}$ and try to find their inverses, we have :

$$\mathbf{A}^{-1} = \frac{1}{2 \times 3 - 0 \times (-1)} \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix} = \frac{1}{6-0} \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$$

giving : $\mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{6} & \frac{1}{3} \end{pmatrix}$.

Since **A has** an **inverse**, it is **non-singular**.

But $\mathbf{B}^{-1} = \dfrac{1}{3 \times 2 - 6 \times 1} \begin{pmatrix} 2 & -6 \\ -1 & 3 \end{pmatrix} = \dfrac{1}{6-6} \begin{pmatrix} 2 & -6 \\ -1 & 3 \end{pmatrix} = \dfrac{1}{0} \begin{pmatrix} 2 & -6 \\ -1 & 3 \end{pmatrix}$.

Since **ad-bc = 0** is undefined,
B has **no inverse** and is, therefore, **singular**.

TYPES OF MATRIX

(1) Row Matrix

This is a matrix which contains **only one row**.

E.g. $\begin{pmatrix} \mathbf{2} & \mathbf{-1} \end{pmatrix}$ is a row matrix.

(2) Column Matrix

This is a matrix which contains **only one column**.

E.g. $\begin{pmatrix} \mathbf{2} \\ \mathbf{-1} \end{pmatrix}$ is a column matrix.

(3) Null or Zero Matrix

This is a matrix which is made up **entirely** of **zero elements**.

E.g. $\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ is a null or zero matrix.

(4) Square Matrix

This is a matrix having the **same number** of **rows** as it has of **columns**.

E.g. $\begin{pmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{-3} & \mathbf{0} \end{pmatrix}$ is a square matrix.

(5) Diagonal Matrix

This is a square matrix whose elements are **all zero except** those on the **main diagonal**.

E.g. $\begin{pmatrix} \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{3} \end{pmatrix}$ is a diagonal matrix.

(**N.B**. The **main diagonal** runs from **upper left** to **lower right**.)

(6) Unit or Identity Matrix

This is a **diagonal matrix** in which **all** the elements on the main diagonal equal **1**.

The **Unit** or **Identity Matrix** is usually denoted by **I**.

E.g. $\mathbf{I} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$.

Multiplication by the **Unit** or **Identity Matrix** is **equivalent** to **multiplication** by **1** in ordinary algebra.

Any matrix **multiplied** by its **inverse** gives the **identity** matrix,

i.e. $\mathbf{AA}^{-1} = \mathbf{I}$.

This compares with ordinary algebra where

$$x \times \frac{1}{x} = 1,$$

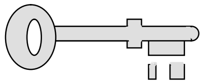
i.e. **any quantity** multiplied by its **reciprocal** is equal to **1**.

EQUALITY OF MATRICES

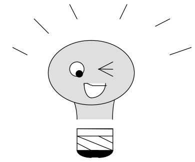
Two matrices are **equal** if, and only if, their **corresponding elements** are **equal**.

E.g. $\begin{pmatrix} \mathbf{a} & \mathbf{3} \\ \mathbf{b} & \mathbf{-1} \end{pmatrix} = \begin{pmatrix} \mathbf{2} & \mathbf{c} \\ \mathbf{3} & \mathbf{d} \end{pmatrix}$

$\Rightarrow$ **a** = **2**, **b** = **3**, **c** = **3** and **d** = **-1**.



KEY POINTS



Matrices

- A matrix which has **m rows** and **n columns** is said to have

  ORDER $\mathbf{m \times n}$.

- **Two** matrices are **equal** if, and only if, their **corresponding elements** are **equal**:

  E.g. $\mathbf{A} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \mathbf{e} & \mathbf{f} \\ \mathbf{g} & \mathbf{h} \end{pmatrix}$

  $\mathbf{A} = \mathbf{B}$

  $\Rightarrow \quad \mathbf{a = e, \quad b = f, \quad c = g, \quad d = h.}$

- Matrices of the **same order** may be **added** or **subtracted**, simply by **adding** or **subtracting corresponding elements** :

  E.g. $\mathbf{A} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \mathbf{e} & \mathbf{f} \\ \mathbf{g} & \mathbf{h} \end{pmatrix}$

  $\Rightarrow \mathbf{A} + \mathbf{B} = \begin{pmatrix} \mathbf{a + e} & \mathbf{b + f} \\ \mathbf{c + g} & \mathbf{d + h} \end{pmatrix}$ and

  $\mathbf{A} - \mathbf{B} = \begin{pmatrix} \mathbf{a - e} & \mathbf{b - f} \\ \mathbf{c - g} & \mathbf{d - h} \end{pmatrix}.$

- Any matrix may be **multiplied** by a **scalar k**, where **k** is a **real** number:

  E.g. $\mathbf{A} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}$

  $\Rightarrow \quad \mathbf{Ka} = \begin{pmatrix} \mathbf{ka} & \mathbf{kb} \\ \mathbf{kc} & \mathbf{kd} \end{pmatrix}$

(Contd. Overleaf)

- A matrix of **order $m \times n$** may be **multiplied** by **another matrix** of **order $n \times p$**.

This means that the **first matrix** must have the **same number** of **columns** as the **second matrix** has of **rows** to make **matrix multiplication possible**.

E.g. $\mathbf{A} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}$, $\quad \mathbf{B} = \begin{pmatrix} \mathbf{e} & \mathbf{f} & \mathbf{g} \\ \mathbf{h} & \mathbf{j} & \mathbf{k} \end{pmatrix}$.

A is of **order 2×2** ; **B** is of **order 2×3**.

Since **A** has the **same number** of **columns** as **B** has of **rows**, it is **possible** to find **AB**.

$$\mathbf{AB} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}\begin{pmatrix} \mathbf{e} & \mathbf{f} & \mathbf{g} \\ \mathbf{h} & \mathbf{j} & \mathbf{k} \end{pmatrix} = \begin{pmatrix} \mathbf{ae+bh} & \mathbf{af+bj} & \mathbf{ag+bk} \\ \mathbf{ce+dh} & \mathbf{cf+dj} & \mathbf{cg+dk} \end{pmatrix}$$

Note that the **resultant matrix AB** has **2 rows** and **3 columns**,
i.e. the **same number** of **rows** as **A** and the **same number** of **columns** as **B**.

This result is important:

*Any matrix of **order $m \times n$** may be **multiplied** by **another** matrix of **order $n \times p$** and the **resultant matrix** is of **order $m \times p$**.*

Note also that **BA** is **not possible** to find since a **2×3** matrix **cannot** be multiplied by a **2×2** matrix.

Clearly, then, $\mathbf{AB} \neq \mathbf{BA}$ in matrix algebra
i.e. **matrices** are **non-commutative** under **multiplication**.

(Contd. Overleaf)

- $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

The **Inverse Matrix**, $A^{-1} = \dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, $ad\text{-}bc \neq 0$.

If **ad-bc = 0**, **A** has **no inverse** and is, therefore, a **singular matrix**.
If **ad-bc** $\neq$ **0**, **A** has an **inverse** and is, therefore,
a **non-singular matrix**.

The **inverse matrix** can be used to **solve simultaneous linear equations**.

E.g. Solve **2x – 3y = -5** and **x – 2y = -4**.

Method:

$$M = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}$$

$$\Rightarrow M^{-1} = \frac{1}{-2 \times 2 - (-3 \times 1)}\begin{pmatrix} -2 & 3 \\ -1 & 2 \end{pmatrix}$$

$$\Rightarrow M^{-1} = \frac{1}{-4 + 3}\begin{pmatrix} -2 & 3 \\ -1 & 2 \end{pmatrix}$$

$$= -1\begin{pmatrix} -2 & 3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}$$

$$M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5 \\ -4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = M^{-1}\begin{pmatrix} -5 \\ -4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} -5 \\ -4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

This means that the solution is given by **x = 2** and **y = 3**.

EXERCISE 1

MATRICES

1. A **matrix** containing **m rows** and **n columns** is said to have **order m × n**. State the **order** of each of the following matrices:

(i) $\mathbf{A} = (2 \quad 1)$ **(ii)** $\mathbf{B} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ **(iii)** $\mathbf{C} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

(iv) $\mathbf{D} = \begin{pmatrix} 4 & 3 & 2 \\ 1 & -1 & 0 \end{pmatrix}$

2. $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ -3 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -2 & -1 \\ 0 & 5 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}$

Calculate the following matrices:

(i) $\mathbf{A} + 2\mathbf{B} - 4\mathbf{C}$ **(ii)** $\frac{1}{2}\mathbf{A} - \frac{3}{2}\mathbf{B} + \frac{1}{4}\mathbf{C}$

(iii) $\sqrt{2}\,\mathbf{A} - 2\sqrt{2}\mathbf{B} + 3\sqrt{2}\mathbf{C}$

3. $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 4 & 0 \\ 3 & -1 & 5 \end{pmatrix}$

Where **possible**, find the following matrices:
(Where **impossible**, state the **reason**).

(i) $\mathbf{AB}$ **(ii)** $\mathbf{BA}$ **(iii)** $\mathbf{A}^2$ **(iv)** $\mathbf{B}^2$ **(v)** $\mathbf{A}^2\mathbf{B}$ **(vi)** $\mathbf{AB}^2$

4. $\mathbf{M} = \begin{pmatrix} 2 & 3 \\ -4 & 1 \end{pmatrix}$

(i) Find $\mathbf{M}^{-1}$, the **inverse matrix** of **M** and, **hence**,

(ii) **solve** the **simultaneous equations** :

$$2x + 3y = 4 \quad . . . \text{ (i)}$$
$$-4x + y = 6 \quad . . . \text{ (ii)}$$

5. **Solve** the following **simultaneous equations** using matrices:

(i)
$$3x + 2y = 4 \quad . . . \text{ (i)}$$
$$-x + y = -3 \quad . . . \text{ (ii)}$$

(ii)
$$-3x - 2y = -1 \quad . . . \text{ (i)}$$
$$5x + y = -3 \quad . . . \text{ (ii)}$$

(iii)
$$3x - 2y = 7 \quad . . . \text{ (i)}$$
$$5x - y = 7 \quad . . . \text{ (ii)}$$

(iv)
$$x - 3y = 9 \quad . . . \text{ (i)}$$
$$5x + y = -3 \quad . . . \text{ (ii)}$$

6. **Determine** whether each matrix below is **singular** or **non-singular**.

Show your working.

E.g. $\begin{pmatrix} \mathbf{2} & \mathbf{6} \\ \mathbf{-1} & \mathbf{-3} \end{pmatrix}$ is **singular** since the **determinant ad - bc** is **equal to 0**, i.e. the **matrix has no inverse**.

(i) **A** = $\begin{pmatrix} \mathbf{2} & \mathbf{3} \\ \mathbf{1} & \mathbf{-4} \end{pmatrix}$

(ii) **B** = $\begin{pmatrix} \mathbf{-1} & \mathbf{3} \\ \mathbf{2} & \mathbf{-6} \end{pmatrix}$

(iii) **C** = $\begin{pmatrix} \mathbf{1} & \mathbf{-2} \\ \mathbf{-3} & \mathbf{6} \end{pmatrix}$

(iv) **D** = $\begin{pmatrix} \mathbf{2} & \mathbf{-8} \\ \mathbf{-1} & \mathbf{4} \end{pmatrix}$

(v) **E** = $\begin{pmatrix} \mathbf{1} & \mathbf{3} \\ \mathbf{2} & \mathbf{4} \end{pmatrix}$

EXERCISE 1 - ANSWERS

1. **(i)** 1×2 **(ii)** 2×1 **(iii)** 2×2 **(iv)** 2×3

2. **(i)** $A + 2B - 4C$

$$= \begin{pmatrix} 1 & 0 \\ -3 & 4 \end{pmatrix} + 2\begin{pmatrix} -2 & -1 \\ 0 & 5 \end{pmatrix} - 4\begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -3 & 4 \end{pmatrix} + \begin{pmatrix} -4 & -2 \\ 0 & 10 \end{pmatrix} - \begin{pmatrix} 8 & 12 \\ -4 & 16 \end{pmatrix}$$

$$= \begin{pmatrix} 1-4-8 & 0-2-12 \\ -3+0+4 & 4+10-16 \end{pmatrix}$$

$$= \begin{pmatrix} -11 & -14 \\ 1 & -2 \end{pmatrix}$$

(ii) $\frac{1}{2}A - \frac{3}{2}B + \frac{1}{4}C$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ -3 & 4 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} -2 & -1 \\ 0 & 5 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{3}{2} & 2 \end{pmatrix} - \begin{pmatrix} -3 & -\frac{3}{2} \\ 0 & \frac{15}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & \frac{3}{4} \\ -\frac{1}{4} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}+3+\frac{1}{2} & 0+\frac{3}{2}+\frac{3}{4} \\ -\frac{3}{2}-0-\frac{1}{4} & 2-\frac{15}{2}+1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & \frac{9}{4} \\ -\frac{7}{4} & -\frac{9}{2} \end{pmatrix}$$

(iii) $\sqrt{2}A - 2\sqrt{2}B + 3\sqrt{2}C$

$$= \sqrt{2}\begin{pmatrix} 1 & 0 \\ -3 & 4 \end{pmatrix} - 2\sqrt{2}\begin{pmatrix} -2 & -1 \\ 0 & 5 \end{pmatrix} + 3\sqrt{2}\begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{2} & 0 \\ -3\sqrt{2} & 4\sqrt{2} \end{pmatrix} - \begin{pmatrix} -4\sqrt{2} & -2\sqrt{2} \\ 0 & 10\sqrt{2} \end{pmatrix} + \begin{pmatrix} 6\sqrt{2} & 9\sqrt{2} \\ -3\sqrt{2} & 12\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{2}+4\sqrt{2}+6\sqrt{2} & 0+2\sqrt{2}+9\sqrt{2} \\ -3\sqrt{2}-0-3\sqrt{2} & 4\sqrt{2}-10\sqrt{2}+12\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 11\sqrt{2} & 11\sqrt{2} \\ -6\sqrt{2} & 6\sqrt{2} \end{pmatrix}$$

3.

(i) $\mathbf{AB} = \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 2 & 4 & 0 \\ 3 & -1 & 5 \end{pmatrix}$

$$= \begin{pmatrix} 2 & 4 & 0 \\ 5 & -11 & 15 \end{pmatrix}$$

(ii) **BA** is **impossible** since **B has order 2×3** and **A has order 2×2**,
i.e B has **not** the same number of columns as A has of rows.

(iii) $\mathbf{A}^2 = \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -8 & 9 \end{pmatrix}$

(iv) $\mathbf{B}^2$ is **impossible** since **B** has **order 2×3**, giving $(2 \times 3) \times (2 \times 3)$.

(v) $\mathbf{A}^2\mathbf{B} = \begin{pmatrix} 1 & 0 \\ -8 & 9 \end{pmatrix}\begin{pmatrix} 2 & 4 & 0 \\ 3 & -1 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 0 \\ 11 & -41 & 45 \end{pmatrix}$

(vi) **Impossible**.

4. **(i)** $\mathbf{M} = \begin{pmatrix} 2 & 3 \\ -4 & 1 \end{pmatrix}$

$$\mathbf{M}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\Rightarrow \frac{1}{2+12}\begin{pmatrix} 1 & -3 \\ 4 & 2 \end{pmatrix}$$

$$= \frac{1}{14}\begin{pmatrix} 1 & -3 \\ 4 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{14} & -\frac{3}{14} \\ \frac{2}{7} & \frac{1}{7} \end{pmatrix}$$

(ii) $\quad 2x + 3y = 4 \ldots$ **(i)**
$\quad -4x + y = 6 \ldots$ **(ii)**

$$\Rightarrow \quad \begin{pmatrix} 2 & 3 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -4 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} \frac{1}{14} & -\frac{3}{14} \\ \frac{2}{7} & \frac{1}{7} \end{pmatrix}\begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$ i.e. $x = -1,\ y = 2$.

5. **(i)** $\quad 3x + 2y = 4 \ldots$ **(i)**
$\quad -x + y = -3 \ldots$ **(ii)**

$$\Rightarrow \quad \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

$$\Rightarrow \quad \frac{1}{3+2}\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

$$\Rightarrow \quad \frac{1}{5}\begin{pmatrix} 10 \\ -5 \end{pmatrix}$$

$$= \quad \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$ i.e. $x = 2,\ y = -1$.

(ii) $\quad -3x - 2y = -1 \ldots$ **(i)**
$\quad 5x + y = -3 \ldots$ **(ii)**

$$\Rightarrow \quad \begin{pmatrix} -3 & -2 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 & -2 \\ 5 & 1 \end{pmatrix}^{-1}\begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

$$\Rightarrow \quad \frac{1}{-3+10}\begin{pmatrix} 1 & 2 \\ -5 & -3 \end{pmatrix}\begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

$$\Rightarrow \quad \frac{1}{7}\begin{pmatrix} -7 \\ 14 \end{pmatrix}$$

$$= \quad \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$ i.e. $x = -1,\ y = 2$.

(iii) $\quad 3x - 2y = 7 \; . \; . \; . \; \text{(i)}$

$\quad 5x - y = 7 \; . \; . \; . \; \text{(ii)}$

$$\Rightarrow \quad \begin{pmatrix} 3 & -2 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 5 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 7 \\ 7 \end{pmatrix}$$

$$\Rightarrow \quad \frac{1}{-3+10} \begin{pmatrix} -1 & 2 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} 7 \\ 7 \end{pmatrix}$$

$$\Rightarrow \quad \frac{1}{7} \begin{pmatrix} 7 \\ -14 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \text{i.e.} \quad x = 1, \; y = -2.$$

(iv) $\quad x - 3y = 9 \; . \; . \; . \; \text{(i)}$

$\quad 5x + y = -3 \; . \; . \; . \; \text{(ii)}$

$$\Rightarrow \quad \begin{pmatrix} 1 & -3 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ -3 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 5 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 9 \\ -3 \end{pmatrix}$$

$$\Rightarrow \quad \frac{1}{1+15} \begin{pmatrix} 1 & 3 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} 9 \\ -3 \end{pmatrix}$$

$$\Rightarrow \quad \frac{1}{16} \begin{pmatrix} 0 \\ -48 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \text{i.e.} \quad x = 0, \; y = -3.$$

6. **(i)** **ad - bc = - 8 + 3 = - 5**

$\therefore$ **A has** an **inverse** and is **non-singular**.

(ii) **ad - bc = 6 - 6 = 0**

$\therefore$ **B** has **no inverse** and is **singular**.

(iii) **ad - bc = 6 - 6 = 0**

$\therefore$ **C** has **no inverse** and is **singular.**

(iv) **ad - bc = 8 - 8 = 0**

$\therefore$ **D** has **no inverse** and is **singular**.

(v) **ad - bc = 4 - 6 = -2**

$\therefore$ **E has** an **inverse** and is **non - singular**.

SECTION 2

TRANSFORMATION MATRICES

In GCSE Section 1 we studied **transformations** in the plane using ***geometrical*** methods.
Now, we shall look at ***matrix*** methods, which do the **same** job and are quick and easy to apply.

The **coordinates** of the **vertices** of the shape undergoing transformation are written as **columns** to form a **matrix** and this matrix is **premultiplied** by the **transformation matrix**.

The **columns** of the **resultant matrix** give the **coordinates** of the **vertices** of the **image** of the **shape** transformed.

Useful transformation matrices

The **transformation matrices** which correspond to the **geometrical transformations** studied earlier are as follows:-

(I) REFLECTIONS

(a) Reflection in the ***x*-axis** (i.e. the line **y = 0**):

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

E.g. Reflect **P (2, 3)** in the ***x*-axis**

$$\begin{array}{ccc} \mathbf{A} & \mathbf{P} & & \mathbf{P_1} \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} 2 \\ 3 \end{pmatrix} & = & \begin{pmatrix} 2 \\ -3 \end{pmatrix} \end{array}$$

Therefore, $\mathbf{P_1}$ **(2, -3)** is the **image** of **P (2, 3)**, following **reflection** in the ***x*-axis**.

(b) Reflection in the **y-axis** (i.e. the line ***x*** **= 0**):

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

E.g. Reflect **P (2, 3)** in the **y-axis**.

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

Therefore, **P_1 (-2, 3)** is the
image of **P (2, 3)**, following **reflection** in the **y-axis**.

(c) **Reflection** in the line **y =** ***x***

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Since **reflection** in the line **y = x** **reverses** the **coordinates** of the point **P (2, 3)**,
it has **(3, 2)** as its **image**.

Premultiplying the point **(2, 3)** by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
gives :

$$\underset{\mathbf{A}}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}\underset{\mathbf{P}}{\begin{pmatrix} 2 \\ 3 \end{pmatrix}} = \underset{\mathbf{P_1}}{\begin{pmatrix} 3 \\ 2 \end{pmatrix}}$$

Therefore, **P_1(3, 2)** is the **image** of
P (2, 3), following **reflection** in the line **y =** ***x***.

(d) **Reflection** in the line **y = -*x*.**

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Since this reflection **reverses** the **coordinates** and **changes** the **signs** of **both**, the point **P (2, 3) reflected** in the line **y = -*x***

has **(-3, -2)** as its **image**.

Premultiplying the point **(2, 3)** by $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ gives:

$$\begin{array}{cc} \mathbf{A} & \mathbf{P} \\ \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} 2 \\ 3 \end{pmatrix} \end{array} = \begin{array}{c} \mathbf{P_1} \\ \begin{pmatrix} -3 \\ -2 \end{pmatrix} \end{array}$$

Therefore, $\mathbf{P_1}$ **(-3, -2)** is the **image** of **P (2, 3)** following **reflection** in the line **y = -*x***.

(II) ROTATIONS

The matrix $\begin{pmatrix} \cos x° & -\sin x° \\ \sin x° & \cos x° \end{pmatrix}$ gives **rotation** through

***x*° anti-clockwise** about the **origin (0, 0)**.

(a) **Rotation** about the origin **(0, 0)** through **+90°** (i.e. **anti-clockwise**) :

$$\begin{pmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{array}{c} \mathbf{P} \\ \begin{pmatrix} 2 \\ 3 \end{pmatrix} \end{array} = \begin{array}{c} \mathbf{P_1} \\ \begin{pmatrix} -3 \\ 2 \end{pmatrix} \end{array}.$$

Therefore, $\mathbf{P_1}$ **(-3, 2)** is the **image** of **P(2, 3)** following **rotation** about the origin **(0, 0)** through **90° anti-clockwise**.

(b) **Rotation** about the origin **(0, 0)** through **-90°** (i.e. **clockwise**) :

Since the **rotation matrix** can be used **only** on **anti-clockwise** rotations, we must use **270°** here, as **-90°** (clockwise) is equivalent to +**270°** (anti-clockwise). Then we have:

$$\begin{pmatrix} \cos 270° & -\sin 270° \\ \sin 270° & \cos 270° \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

E.g. Rotate **P (2, 3)** about the origin **(0, 0)** through **-90°** (i.e. **+270°**)

Premultiplying the point **(2, 3)** by $\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-1} & \mathbf{0} \end{pmatrix}$ gives:

$$\begin{array}{ccc} & \mathbf{P} & & \mathbf{P_1} \\ \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-1} & \mathbf{0} \end{pmatrix} & \begin{pmatrix} \mathbf{2} \\ \mathbf{3} \end{pmatrix} & = & \begin{pmatrix} \mathbf{3} \\ \mathbf{-2} \end{pmatrix} \end{array}$$

Therefore, $\mathbf{P_1}$ **(3, -2)** is the **image** of **P (2, 3)** following **rotation** about the origin **(0, 0)** through **270°** anti-clockwise.

(c) **Rotation** about the origin (**0, 0)** through ±**180°**:

$$\begin{pmatrix} \mathbf{\cos 180°} & \mathbf{-\sin 180°} \\ \mathbf{\sin 180°} & \mathbf{\cos 180°} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix}$$

E.g. Rotate the point **P (2, 3)** about the origin **(0, 0)** through **180°**.

Premultiplying the point **(2, 3)** by $\begin{pmatrix} \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix}$ gives:

$$\begin{array}{ccc} & \mathbf{P} & & \mathbf{P_1} \\ \begin{pmatrix} \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix} & \begin{pmatrix} \mathbf{2} \\ \mathbf{3} \end{pmatrix} & = & \begin{pmatrix} \mathbf{-2} \\ \mathbf{-3} \end{pmatrix} \end{array}$$

Therefore, $\mathbf{P}_1$ **(-2, -3)** is the **image** of **P (2, 3)**

following **rotation** about the origin **(0, 0)**

through **180°** (**clockwise** or **anti-clockwise**).

(iii) ENLARGEMENT (or REDUCTION) USING A SCALE FACTOR K.

(a) **Enlargement** (or **reduction**), **centre** the origin **(0, 0)**, using a **scale factor k** :

$$\begin{pmatrix} \mathbf{k} & \mathbf{0} \\ \mathbf{0} & \mathbf{k} \end{pmatrix}$$

E.g. The triangle **ABC** has vertices (**2, 3), (-1, 4)** and **(1, -2)** respectively.

Find the vertices of $\mathbf{A_1 B_1 C_1}$**,** the **image** of **ABC** following an **enlargement**, **centre (0, 0),** using **scale factor** $\frac{3}{2}$.

$$\begin{array}{cc} & \begin{array}{ccc} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{array} \\ \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix} & \begin{pmatrix} 2 & -1 & 1 \\ 3 & 4 & -2 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} \mathbf{A_1} & \mathbf{B_1} & \mathbf{C_1} \end{array} \\ \begin{pmatrix} 3 & -\frac{3}{2} & \frac{3}{2} \\ \frac{9}{2} & 6 & -3 \end{pmatrix} \end{array}$$

Therefore the triangle $\mathbf{A_1 B_1 C_1}$, the **image** of triangle **ABC**, following **enlargement**, using **centre (0,0)** and **scale factor** $\frac{3}{2}$ has **vertices**

(**3,** $\frac{9}{2}$**), (** $-\frac{3}{2}$**, 6**) and **(**$\frac{3}{2}$**, -3)** respectively.

Enlargement (or **reduction),** using a **scale factor k,** where the **centre** is **outside** the **origin (0, 0),** is made up of an **enlargement, scale factor k, centre** the **origin (0, 0)** and a **translation.**

The quickest way to do this enlargement, (for **k positive**), is to **premultiply** each vertex by $\begin{pmatrix} \mathbf{k} & \mathbf{0} \\ \mathbf{0} & \mathbf{k} \end{pmatrix}$ and then **add** on the **translation** that would **move the centre** of the enlargement **to the origin**.

E.g. The triangle **ABC** with vertices **(1, 1), (3, 2)** and **(2, -1)** respectively is **enlarged** by a **scale factor** of **2**, **centre (-1, -2)**.

Find the vertices of $\mathbf{A_1 B_1 C_1}$, the image of **ABC** under this enlargement.

Clearly, a **translation** of $\begin{pmatrix} \mathbf{1} \\ \mathbf{2} \end{pmatrix}$ would move the **centre (-1, -2)** to the **origin (0, 0).**

(Simply **change** the **signs** of **both coordinates** of the **centre**).

Enlargement of **ABC**, using a **scale factor** of **2**, **centre** the **origin (0, 0)** gives:

$$\begin{pmatrix} \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{2} \end{pmatrix} \begin{array}{c} \begin{array}{ccc} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{array} \\ \begin{pmatrix} \mathbf{1} & \mathbf{3} & \mathbf{2} \\ \mathbf{1} & \mathbf{2} & \mathbf{-1} \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} \mathbf{A_1} & \mathbf{B_1} & \mathbf{C_1} \end{array} \\ \begin{pmatrix} \mathbf{2} & \mathbf{6} & \mathbf{4} \\ \mathbf{2} & \mathbf{4} & \mathbf{-2} \end{pmatrix} \end{array}$$

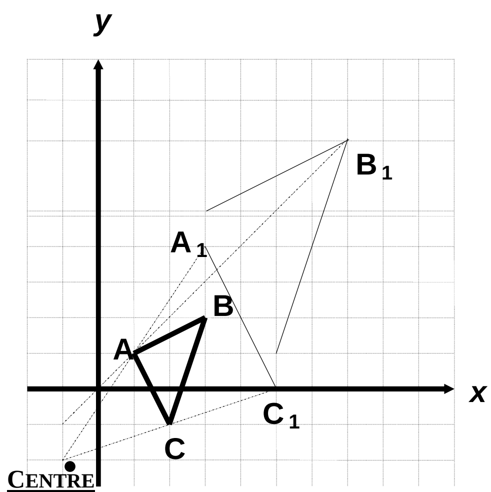
Therefore, $\mathbf{A}_1 = \begin{pmatrix} \mathbf{2} \\ \mathbf{2} \end{pmatrix} + \textbf{Translation} \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \end{pmatrix} = \begin{pmatrix} \mathbf{3} \\ \mathbf{4} \end{pmatrix}$

$\mathbf{B}_1 = \begin{pmatrix} \mathbf{6} \\ \mathbf{4} \end{pmatrix} + \textbf{Translation} \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \end{pmatrix} = \begin{pmatrix} \mathbf{7} \\ \mathbf{6} \end{pmatrix}$

$\mathbf{C}_1 = \begin{pmatrix} \mathbf{4} \\ -\mathbf{2} \end{pmatrix} + \textbf{Translation} \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \end{pmatrix} = \begin{pmatrix} \mathbf{5} \\ \mathbf{0} \end{pmatrix}.$

So $\mathbf{A}_1\mathbf{B}_1\mathbf{C}_1$ has **vertices (3, 4)**, **(7, 6)** and **(5, 0)** respectively.

(See **Fig**. below).



(iv) TRANSLATION THROUGH $\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}$

To **translate** a shape by a displacement vector $\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}$ it is convenient to form a **coordinate matrix** from the **vertices** of the shape and **add on** the displacement vector $\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}$ to **each column**.

E.g. A triangle **ABC** with vertices **(1, 1), (3, 2)** and **(2, -1)** is **translated** by the displacement vector $\begin{pmatrix} \mathbf{-1} \\ \mathbf{3} \end{pmatrix}$.

Find the **vertices** of the triangle $\mathbf{A_1B_1C_1}$, the **image** of **ABC** under this **translation**.

$$\begin{array}{ccccc} \mathbf{A\ \ B\ \ C} & & \textbf{Translation} & & \mathbf{A_1B_1C_1} \\ \begin{pmatrix} \mathbf{1} & \mathbf{3} & \mathbf{2} \\ \mathbf{1} & \mathbf{2} & \mathbf{-1} \end{pmatrix} & + & \begin{pmatrix} \mathbf{-1} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{3} & \mathbf{3} & \mathbf{3} \end{pmatrix} & + & \begin{pmatrix} \mathbf{0} & \mathbf{2} & \mathbf{1} \\ \mathbf{4} & \mathbf{5} & \mathbf{2} \end{pmatrix} \end{array}$$

Therefore the **vertices** of the triangle $\mathbf{A_1B_1C_1}$ are **(0, 4), (2, 5)** and **(1, 2)** respectively.

TO FIND THE TRANSFORMATION MATRIX

It is possible to **find** the **transformation matrix** that will **map** a **point** (or set of vertices) from **one position** to **another**, by setting up a transformation matrix and proceeding in the normal way.

(i) If only **one point** is being **transformed**, set up a **diagonal matrix** for the transformation and **premultiply** the **coordinates** of the point by it.

E.g. The point **P (2, 3)** is mapped to $\mathbf{P_1}$ **(4, 7)** by a transformation. Find the **transformation matrix T**.

Let **T** $= \begin{pmatrix} \mathbf{a} & \mathbf{0} \\ \mathbf{0} & \mathbf{b} \end{pmatrix}$

Then $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$

$\Rightarrow \quad \begin{pmatrix} a \times 2 + 0 \times 3 \\ 0 \times 2 + b \times 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$

$\Rightarrow \quad \begin{pmatrix} 2a \\ 3b \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$

By **equality** of **matrices** :

$\mathbf{2a = 4 \quad \Rightarrow \quad a = 2}$

and $\mathbf{3b = 7 \quad \Rightarrow \quad b = \frac{7}{3}}$.

$\therefore$ **T** $= \begin{pmatrix} 2 & 0 \\ 0 & \frac{7}{3} \end{pmatrix}$ is the **transformation matrix**,

which will map **P (2, 3)** to $\mathbf{P_1}$ **(4, 7)**.

(ii) If **two** or **more points** are being transformed, a diagonal matrix is insufficient. In this case it is necessary to set up a **full matrix**.

E.g. The line **PQ** (with **P = (2, 3)** and **Q = (4, -1)**) is mapped onto $\mathbf{P_1Q_1}$ (with $\mathbf{P_1}$ = **(4, 7)** and $\mathbf{Q_1}$ = **(8, -7)**). Find the **transformation matrix T**.

Let **T** $= \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{array}{cccc} & \mathbf{T} & \mathbf{P\ \ Q} & & \mathbf{P_1\ \ Q_1} \\ \text{Then} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix} & = & \begin{pmatrix} 4 & 8 \\ 7 & -7 \end{pmatrix} \end{array}$$

By **equality** of **matrices**:

	2a + 3b =	**4**	and	**2c + 3d**	**= 7…(i)**
	4a - b =	**8**		**4c - d**	**= -7…(ii)**
(ii) × 3…	**12a - 3b =**	**24**	**(ii) × 3 . . .**	**12c - 3d**	**=-21…(iii)**
(i) + (iii)…	**14a =**	**28**	**(i) + (iii)...**	**14c**	**=-14**
$\therefore$	**a =**	**2**		**c**	**= - 1**
	b =	**0**		**d**	**= 3**

Therefore **T** $= \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$ is the **transformation matrix**

which will map **PQ** to $\mathbf{P_1Q_1}$.

DOUBLE TRANSFORMATION MATRICES

A **double transformation** is made up of **one transformation** (e.g. reflection in the *x*-axis) **followed** by a **second transformation** (e.g. enlargement, centre (0, 0), scale factor 2).

The **double** transformation may be performed by **one** operation, if we **multiply** together the **two matrices** describing these transformations, thereby forming a **single transformation matrix**. We then multiply the **coordinate matrix** of the **point** (or set of points forming the vertices of the shape to be transformed) by this matrix.

It is important to remember that, in order to form the **single transformation matrix**, the **first** transformation matrix must be **pre-multiplied** by the **second** transformation matrix.

This means that, if **M** is the **first** and **P** is the **second** of the two transformation matrices to be used in transforming a point or a shape, the **single transformation matrix** combining **M** and **P** is **PM**, <u>*not*</u> **MP**.

E.g. A triangle **ABC** with vertices **(1, 2), (2, -3),** and **(-1, 1)** respectively is given a **double** transformation under $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ followed by $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

Find the vertices of $\mathbf{A_1B_1C_1}$, the **image** of **ABC** following this **double transformation**.

Let **M** $= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ [This matrix gives **reflection** in the ***x*-axis**]

and **P** $= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

[This matrix gives **enlargement** of **ABC**, **centre (0,0)**, **scale factor 2**]

The **single transformation matrix R** is, **PM**:

$$\therefore \quad \mathbf{R} = \begin{array}{cc} \mathbf{P} & \mathbf{M} \\ \begin{pmatrix} \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{2} \end{pmatrix} & \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix} \end{array} = \begin{pmatrix} \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{-2} \end{pmatrix}$$

Writing the **vertices** of **ABC** as a **coordinate matrix** and **premultiplying** it by **R** we have :

$$\begin{array}{cc} \mathbf{R} & \begin{array}{ccc} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{array} \\ \begin{pmatrix} \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{-2} \end{pmatrix} & \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{2} & \mathbf{-3} & \mathbf{1} \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} \mathbf{A_1} & \mathbf{B_1} & \mathbf{C_1} \end{array} \\ \begin{pmatrix} \mathbf{2} & \mathbf{4} & \mathbf{-2} \\ \mathbf{-4} & \mathbf{6} & \mathbf{-2} \end{pmatrix} \end{array}$$

Therefore the **vertices** of $\mathbf{A_1 B_1 C_1}$ are (**2, -4), (4, 6)** and **(-2, -2)** respectively, following the **double** transformation of **ABC** under

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix} \textbf{ followed} \text{ by } \begin{pmatrix} \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{2} \end{pmatrix}.$$

INVERSE TRANSFORMATION MATRICES

The **inverse** of a transformation matrix will **transform** the **image** of a **point** (or of a shape) **back** to its **original position**.

E.g. If the **transformation matrix T** $= \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$

operates on a **triangle ABC**
with vertices **(1, -1), (-2, 3)** and **(3, 4)** respectively,
the **image** $\mathbf{A_1 B_1 C_1}$ of **ABC** under this transformation has **vertices (-1, -7), (4, 18)** and **(11, 7)** respectively.
Then the **inverse matrix** $\mathbf{T^{-1}}$ will
map $\mathbf{A_1 B_1 C_1}$ **back to ABC**.

We have :

$$\underset{\mathbf{T}}{\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}} \begin{array}{ccc} \mathbf{A} & \mathbf{B} & \mathbf{C} \\ \end{array} \!\!\!\!\!\!\!\! \begin{pmatrix} 1 & -2 & 3 \\ -1 & 3 & 4 \end{pmatrix} = \begin{array}{c} \mathbf{A_1}\ \mathbf{B_1}\ \mathbf{C_1} \\ \begin{pmatrix} -1 & 4 & 11 \\ -7 & 18 & 7 \end{pmatrix} \end{array}$$

$$\Rightarrow \quad \begin{array}{c} \mathbf{A}\ \ \mathbf{B}\ \ \mathbf{C} \\ \begin{pmatrix} 1 & -2 & 3 \\ -1 & 3 & 4 \end{pmatrix} \end{array} = \begin{array}{c} \mathbf{T^{-1}} \\ \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}^{-1} \end{array} \begin{array}{c} \mathbf{A_1}\ \mathbf{B_1}\ \mathbf{C_1} \\ \begin{pmatrix} -1 & 4 & 11 \\ -7 & 18 & 7 \end{pmatrix} \end{array}$$

$$= \frac{1}{4-(-6)} \begin{pmatrix} 4 & -2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} -1 & 4 & 11 \\ -7 & 18 & 7 \end{pmatrix}$$

$$= \frac{1}{10} \begin{pmatrix} 10 & -20 & 30 \\ -10 & 30 & 40 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 & 3 \\ -1 & 3 & 4 \end{pmatrix}$$

So, clearly $\quad \mathbf{ABC} \quad = \quad \mathbf{T^{-1} \times (A_1\ B_1\ C_1)}$.

Inverse Reflection Matrices

Since the **inverse** process to **reflection** in a **line** is **reflection again** in the **same line**, the **inverse matrix** of **any reflection matrix** is **itself**.

For example, a point **P (2, 3)** reflected in the **x-axis** has $\mathbf{P_1}$ **(2, -3)** as its image and $\mathbf{P_1}$ **(2, -3)** reflected in the ***x*-axis** has **P (2, 3)** as its **image**.

We shall look at the **reflection matrices** studied earlier:

(a) $\mathbf{A} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix}$ has **inverse** $\mathbf{A^{-1}}$

$$\begin{aligned} \mathbf{A^{-1}} &= \frac{\mathbf{1}}{\mathbf{-1-0}} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix} \\ &= \mathbf{-1} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix} \end{aligned}$$

i.e. $\mathbf{A^{-1}} = \mathbf{A}$.

(b) $\mathbf{B} = \begin{pmatrix} \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$ has inverse $\mathbf{B^{-1}}$

$$\begin{aligned} \mathbf{B^{-1}} &= \frac{\mathbf{1}}{\mathbf{-1-0}} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix} \\ &= \mathbf{-1} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \end{aligned}$$

i.e. $\mathbf{B^{-1}} = \mathbf{B}$.

(c) $\mathbf{C} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$ has inverse $\mathbf{C^{-1}}$

$$\begin{aligned} \mathbf{C^{-1}} &= \frac{\mathbf{1}}{\mathbf{0-1}} \begin{pmatrix} \mathbf{0} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{0} \end{pmatrix} \\ &= \mathbf{-1} \begin{pmatrix} \mathbf{0} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{0} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \end{aligned}$$

i.e. $\mathbf{C^{-1}} = \mathbf{C}$.

(d) $\mathbf{D} = \begin{pmatrix} \mathbf{0} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{0} \end{pmatrix}$ has inverse $\mathbf{D^{-1}}$

$$\begin{aligned} \mathbf{D^{-1}} &= \frac{\mathbf{1}}{\mathbf{0-1}} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \\ &= \mathbf{-1} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{0} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{0} \end{pmatrix} \end{aligned}$$

i.e. $\mathbf{D^{-1}} = \mathbf{D}$.

Inverse Rotation Matrices

Clearly, the **inverse process** to **rotation** through x° in **one direction** is **rotation** through x° in the **opposite direction**.

For example, a point **P(2, 3)** rotated through **+90°** (i.e. **anti-clockwise**) about the **origin (0, 0)** has $\mathbf{P_1}$ **(-3, 2)** as its **image** and $\mathbf{P_1}$ **(-3, 2)** rotated through **-90°** (i.e. **clockwise**) about the **origin (0, 0)** has **P (2, 3)** as its **image**.

We shall look at the **rotation matrices** studied earlier:

(a) $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has **inverse** $\mathbf{A}^{-1} = \dfrac{1}{0-(-1)}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

We found earlier that $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the matrix which gives the **rotation** through **+90°** about the **origin (0, 0)** and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ gives **rotation** through **-90°** about the origin **(0, 0)**.

E.g. A point **P (2, 3)** is rotated through **+90°** about the **origin (0, 0)** to give $\mathbf{P_1}$ **(-3, 2)** thus :

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\underset{\mathbf{P}}{\begin{pmatrix} 2 \\ 3 \end{pmatrix}} = \underset{\mathbf{P_1}}{\begin{pmatrix} -3 \\ 2 \end{pmatrix}}$$

Now if we **rotate** $\mathbf{P_1}$ **(-3, 2)** through **-90°** (equivalent to **+270°**, remember!) $\mathbf{P_1}$ **(-3, 2)** is mapped **back** to **P (2, 3)**:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\underset{\mathbf{P_1}}{\begin{pmatrix} -3 \\ 2 \end{pmatrix}} = \underset{\mathbf{P}}{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}$$

(b) $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ has **inverse** $\mathbf{B}^{-1} = \dfrac{1}{0-(-1)}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Since $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ gives **rotation** through **-90°** (or **+270°**) about the **origin (0, 0)**, the **inverse matrix** must give **rotation** through **+90°** about the **origin (0, 0)**, and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *does* give this.

(c) $\mathbf{C} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ has **inverse** $\mathbf{C}^{-1}$

$$\mathbf{C}^{-1} = \frac{1}{1-0}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

i.e. $\mathbf{C}^{-1} = \mathbf{C}$.

As we found earlier, the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ gives **rotation** through **±180°**, the **inverse** of this **rotation** is **itself**.

E.g. A point **P (2, 3)** is **rotated** through **180°** about the **origin (0, 0)** to give $\mathbf{P_1}$ **(-2, -3)** thus :

$$\begin{array}{ccc} & \mathbf{P} & \mathbf{P_1} \\ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} 2 \\ 3 \end{pmatrix} = & \begin{pmatrix} -2 \\ -3 \end{pmatrix} \end{array}$$

Now if we **rotate** $\mathbf{P_1}$ **(-2, -3)** through **180°**, $\mathbf{P_1}$ **(-2, -3)** is mapped **back** to **P(2, 3)**:

$$\begin{array}{ccc} & \mathbf{P_1} & \mathbf{P} \\ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} -2 \\ -3 \end{pmatrix} = & \begin{pmatrix} 2 \\ 3 \end{pmatrix}. \end{array}$$

Inverse Enlargement (or Reduction) Matrices, centre the origin (0, 0), scale factor k.

The **inverse** of $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ is

$$\frac{1}{k^2 - 0}\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} = \frac{1}{k^2}\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} = \begin{pmatrix} \frac{1}{k} & 0 \\ 0 & \frac{1}{k} \end{pmatrix}.$$

E.g. A **triangle ABC** with **vertices (-1, 1)**, **(2, -1)** and **(-2, 3)** respectively is **enlarged** by a **scale factor** of **2**, **centre** the **origin (0, 0)** to give $\mathbf{A_1B_1C_1}$ with vertices **(-2, 2)**, **(4, -2)** and **(-4, 6)** respectively.

Thus:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{array}{c} \begin{array}{ccc} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{array} \\ \begin{pmatrix} -1 & 2 & -2 \\ 1 & -1 & 3 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} \mathbf{A_1} & \mathbf{B_1} & \mathbf{C_1} \end{array} \\ \begin{pmatrix} -2 & 4 & -4 \\ 2 & -2 & 6 \end{pmatrix} \end{array}$$

Now if we **reduce $A_1B_1C_1$** by a **scale factor $\frac{1}{2}$**, **centre** the **origin (0, 0)**, **$A_1B_1C_1$** is mapped **back** to **ABC** thus :

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{array}{c} \begin{array}{ccc} \mathbf{A_1} & \mathbf{B_1} & \mathbf{C_1} \end{array} \\ \begin{pmatrix} -2 & 4 & -4 \\ 2 & -2 & 6 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{array} \\ \begin{pmatrix} -1 & 2 & -2 \\ 1 & -1 & 3 \end{pmatrix} \end{array}.$$

InverseTranslations

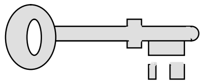
The **inverse translation** of $\begin{pmatrix} x \\ y \end{pmatrix}$ is $\begin{pmatrix} -x \\ -y \end{pmatrix}$.
Clearly, **changing** the **signs** of the **components** is all that is required.

E.g. A point **P = (2,3)** is **transformed** by a **translation**
T = $\begin{pmatrix} +3 \\ -2 \end{pmatrix}$ to give $\mathbf{P_1}$ = **(5,1)** thus:

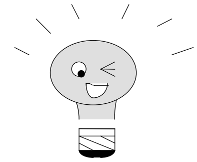
$$\begin{array}{c} \mathbf{P} \\ \begin{pmatrix} 2 \\ 3 \end{pmatrix} \end{array} + \begin{array}{c} \mathbf{T} \\ \begin{pmatrix} +3 \\ -2 \end{pmatrix} \end{array} = \begin{array}{c} \mathbf{P_1} \\ \begin{pmatrix} 5 \\ 1 \end{pmatrix} \end{array}.$$

Now, if $\mathbf{P_1}$ **(5,1)** is **transformed** by a **translation T_1 =** $\begin{pmatrix} -3 \\ +2 \end{pmatrix}$,
P is mapped **back** to **P (2,3)** thus:

$$\begin{array}{c} \mathbf{P_1} \\ \begin{pmatrix} 5 \\ 1 \end{pmatrix} \end{array} + \begin{array}{c} \mathbf{T_1} \\ \begin{pmatrix} -3 \\ +2 \end{pmatrix} \end{array} = \begin{array}{c} \mathbf{P} \\ \begin{pmatrix} 2 \\ 3 \end{pmatrix} \end{array}.$$



KEY POINTS



Transformation Matrices

- To **transform** a **point** (or set of **vertices** of a shape), set up a **coordinate matrix** in which the **coordinates** of the **point(s)** form a **column** (or columns) and **PREMULTIPLY** by the **transformation matrix**.

  Generally, if **T** = $\begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}$ is a **transformation matrix** which **maps** a **point P(*x*,y)** to $\mathbf{P_1}$, we have:

$$\begin{array}{ccc} \mathbf{T} \quad \mathbf{P} & & \mathbf{P_1} \\ \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} & = & \begin{pmatrix} \mathbf{ax} + \mathbf{by} \\ \mathbf{cx} + \mathbf{dy} \end{pmatrix}, \end{array}$$

  giving $\mathbf{P_1}$ **= (a*x* + by, c*x* + dy)**.

- To **find** the **transformation matrix** which will map **a point** from **one position** to **another**, set up a **diagonal matrix** $\begin{pmatrix} \mathbf{a} & \mathbf{0} \\ \mathbf{0} & \mathbf{b} \end{pmatrix}$, **premultiply** the **point** by it and **solve** for **a** and **b**.

- To **find** the **transformation matrix** which will map **two** (or more) **points from one position** to **another**, set up a **full matrix** $\begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}$, **premultiply** the **coordinate matrix** by it and **solve simultaneous equations** to determine the values of **a**, **b**, **c** and **d**.

- **Common Transformation Matrices:**

$\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix}$ gives **reflection** in the ***x*-axis** (i.e. the line **y = 0**).

$\begin{pmatrix} \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$ gives **reflection** in the **y-axis** (i.e. the line ***x* = 0**).

$\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$ gives **reflection** in the line **y = *x***.

$\begin{pmatrix} \mathbf{0} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{0} \end{pmatrix}$ gives **reflection** in the line **y = -*x***.

$\begin{pmatrix} \mathbf{\cos x°} & \mathbf{-\sin x°} \\ \mathbf{\sin x°} & \mathbf{\cos x°} \end{pmatrix}$ gives **rotation** through ***x*° anti-clockwise** about the **origin (0,0)**.

$\begin{pmatrix} \mathbf{0} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$ gives **rotation** through **90° anti-clockwise** about the **origin (0,0)**.

$\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-1} & \mathbf{0} \end{pmatrix}$ gives **rotation** through **90° clockwise** about the **origin (0,0)**.

$\begin{pmatrix} \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix}$ gives **rotation** through **180°** about the **origin (0,0)**.

This transformation is sometimes called '**reflection in the origin**' – notice that it is equivalent to the **double transformation** 'reflection in one coordinate axis, followed by reflection in the other coordinate axis'.

$\begin{pmatrix} \mathbf{k} & \mathbf{0} \\ \mathbf{0} & \mathbf{k} \end{pmatrix}$ gives **enlargement** (or **reduction**),

centre the **origin (0,0)**, using **scale factor k**.

- A **double transformation matrix** can be formed by **multiplying two transformation matrices** together.
  It is important to remember that the matrix for the **first transformation** must be **premultiplied** by the matrix for the **second transformation**.

  Eg. The point **P(2,3)** is transformed to $\mathbf{P_1}$ by a **rotation** through **90° anti-clockwise** about the **origin (0,0)**, followed by **reflection** in the line $\mathbf{y = \mathit{x}}$. Find the coordinates of $\mathbf{P_1}$ using the **double transformation matrix**.

  Since $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ gives **rotation** through **90° anti-clockwise** about the **origin (0,0)**, it must be **premultiplied** by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,
  which gives **reflection** in the line $\mathbf{y = \mathit{x}}$. The **double transformation matrix** is therefore:

  $$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

  (Notice that this double transformation is equivalent to reflection in the *x*-axis.)

  $$\begin{array}{ccc} \mathbf{A} \quad \mathbf{P} & & \mathbf{P_1} \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} & = & \begin{pmatrix} 2 \\ -3 \end{pmatrix}, \end{array}$$

  giving the coordinates of $\mathbf{P_1}$ as **(2, -3)**.

- The **inverse** of a transformation matrix will map the **image** of a point (or a shape) **back** to its **original position**.

  Generally, if $\mathbf{T} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ **transforms** a point **P** to $\mathbf{P_1}$, then the

  Inverse Matrix, $\mathbf{T^{-1}} = \dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, $\mathbf{ad\text{-}bc \neq 0}$

  will map $\mathbf{P_1}$ back to **P**.

EXERCISE 2

TRANSFORMATION MATRICES

1. The **triangle ABC** has vertices **(-2, 1), (0, 3)** and **(1, 0)** respectively.

(a) Under a **reflection** in the line **y = 0**, the **image** of triangle **ABC** is $\mathbf{A_1B_1C_1}$.

(i) Draw triangles **ABC** and $\mathbf{A_1B_1C_1}$ on squared paper.

(ii) Write down the **2 × 2 matrix** associated with this **transformation**.

(b) Under the **transformation** defined by the matrix

$$\mathbf{M} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

the **image** of triangle $\mathbf{A_1B_1C_1}$ is $\mathbf{A_2B_2C_2}$.

(i) Find the **vertices** of **triangle** $\mathbf{A_2B_2C_2}$ and **draw** it on your diagram.

(ii) **Describe** in **geometrical terms** the **transformation** defined by **M**.

(c) Find the **2 × 2 matrix N** under which the **image** of **triangle** $\mathbf{A_2B_2C_2}$ is **triangle ABC**.
Describe in **geometrical terms** the **transformation** defined by **N**.

2. The **square ABCD** has vertices **(2, 1)**, **(5, 4)**, **(2, 7)** and **(-1, 4)** respectively.

(a) Under a **reflection** in the line **y = -*x***, the **image** of square **ABCD** is $\mathbf{A_1B_1C_1D_1}$.

(i) **Draw** squares **ABCD** and $\mathbf{A_1B_1C_1D_1}$ on squared paper.

(ii) Write down the **2 × 2 matrix** associated with this **transformation**.

(b) Under the **tranformation** defined by the matrix:

$$\mathbf{R} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

the **image** of $\mathbf{A_1B_1C_1D_1}$ is $\mathbf{A_2B_2C_2D_2}$.

(i) Find the **vertices** of square $\mathbf{A_2B_2C_2D_2}$ and **draw** it on your diagram.

(ii) **Describe** in **geometrical terms** the **transformation** defined by **R**.

(c) The **matrix D** maps $\mathbf{A_2B_2C_2D_2}$ back to **ABCD**. Find **D**.
Describe in **geometrical terms** the **transformation** defined by **D**.

3. $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -2 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$

(a) The **line PQ**, where **P = (2, 3)** and **Q = (1, -2)**, is **transformed** to $\mathbf{P_1Q_1}$ by the **transformation matrix A**. Find $\mathbf{P_1}$ and $\mathbf{Q_1}$.

(b) The **line $P_1 Q_1$** is **transformed** to **$P_2 Q_2$** by the **transformation matrix B**. Find **P_2** and **Q_2**.

(c) Find the **single transformation matrix** that is **equivalent** to the **double transformation**, **A followed** by **B**.
Check this by **transforming PQ** to **$P_2 Q_2$**, using your **single transformation matrix**.

4. The **transformation matrix**

$\mathbf{T} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}$ maps the **line AB**, where **A = (1, -3)** and **B = (-2, 1)**, to the **line $A_1 B_1$**, with **A_1 = (1, 3)** and **B_1 = (2, 1)**.

Find **a**, **b**, **c** and **d**, using **simultaneous equations** and, **hence**, write out the **matrix T**.

EXERCISE 2 - ANSWERS

TRANSFORMATION MATRICES

1.

(a) **(i)** See diagram.

(ii) $\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix}$

(b) **(i)**

$$\begin{array}{ccc} \mathbf{M} & \mathbf{A_1 \; B_1 \; C_1} & & \mathbf{A_2 \; B_2 \; C_2} \\ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} -2 & 0 & 1 \\ -1 & -3 & 0 \end{pmatrix} & = & \begin{pmatrix} 2 & 0 & -1 \\ -1 & -3 & 0 \end{pmatrix} \end{array}$$

See $\mathbf{A_2 B_2 C_2}$ on diagram - page 69.

(ii) **Reflection** in the line $\boldsymbol{x} = \mathbf{0}$, i.e. the **y - axis**.

(c) Since **N** is the **matrix** which is found from **reflection** in the **y - axis**, followed by **reflection** in the ***x*** **- axis**, we have :

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \mathbf{N}.$$

Reflection in the **origin**.

2.

(a) **(i)** See diagram.

(ii) $\begin{pmatrix} \mathbf{0} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{0} \end{pmatrix}$

(b) **(i)**

$$\begin{array}{ccc} \mathbf{R} & \mathbf{A_1 \; B_1 \; C_1 \; D_1} & & \mathbf{A_2 \; B_2 \; C_2 \; D_2} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} -1 & -4 & -7 & -4 \\ -2 & -5 & -2 & 1 \end{pmatrix} & = & \begin{pmatrix} -2 & -5 & -2 & 1 \\ -1 & -4 & -7 & -4 \end{pmatrix} \end{array}$$

See $\mathbf{A_2 B_2 C_2 D_2}$ on diagram.

(ii) **Reflection** in the line $\mathbf{y} = \boldsymbol{x}$.

(c) Since **D** is the **matrix** which is formed from **reflection** in $\mathbf{y} = \boldsymbol{x}$, **followed** by **reflection** in $\mathbf{y} = \boldsymbol{-x}$, we have :

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Reflection in the **origin**.

3. **(a)**

$$\begin{array}{cccc} \mathbf{A} & \mathbf{P}\quad\mathbf{Q} & & \mathbf{P_1}\quad\mathbf{Q_1} \\ \begin{pmatrix} 1 & 2 \\ -2 & 0 \end{pmatrix} & \begin{pmatrix} 2 & 1 \\ 3 & -2 \end{pmatrix} & = & \begin{pmatrix} 8 & -3 \\ -4 & -2 \end{pmatrix} \end{array}$$

Therefore, $\mathbf{P_1} = \mathbf{(8, -4)}$ and $\mathbf{Q_1} = \mathbf{(-3, -2)}$.

(b)

$$\begin{array}{cccc} \mathbf{B} & \mathbf{P_1}\quad\mathbf{Q_1} & & \mathbf{P_2}\quad\mathbf{Q_2} \\ \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} & \begin{pmatrix} 8 & -3 \\ -4 & -2 \end{pmatrix} & = & \begin{pmatrix} 12 & -8 \\ -20 & -3 \end{pmatrix} \end{array}$$

Therefore, $\mathbf{P_2} = \mathbf{(12, -20)}$ and $\mathbf{Q_2} = \mathbf{(-8, -3)}$.

(c) $\mathbf{BA} = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -2 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 4 \\ -7 & -2 \end{pmatrix}$$

Check: $\begin{pmatrix} 0 & 4 \\ -7 & -2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 12 & -8 \\ -20 & -3 \end{pmatrix}$

Q.E.D.

4.

$$\begin{array}{cccc} \mathbf{T} & \mathbf{A}\quad\mathbf{B} & & \mathbf{A_1}\quad\mathbf{B_1} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \begin{pmatrix} 1 & -2 \\ -3 & 1 \end{pmatrix} & = & \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \end{array}$$

$$\begin{pmatrix} a - 3b & -2a + b \\ c - 3d & -2c + d \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$$

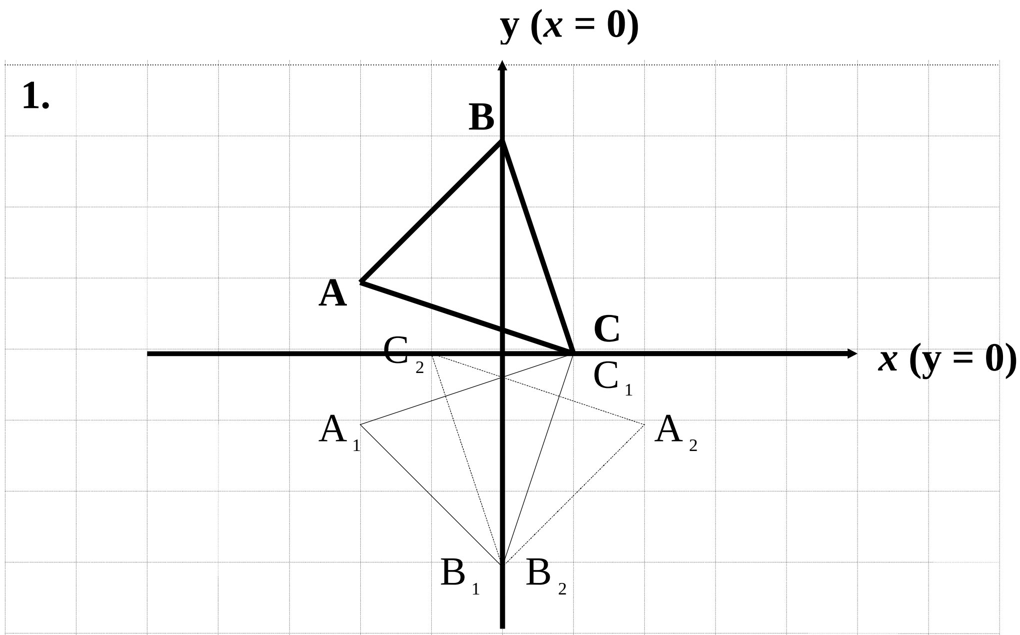
By **equality** of **matrices**:

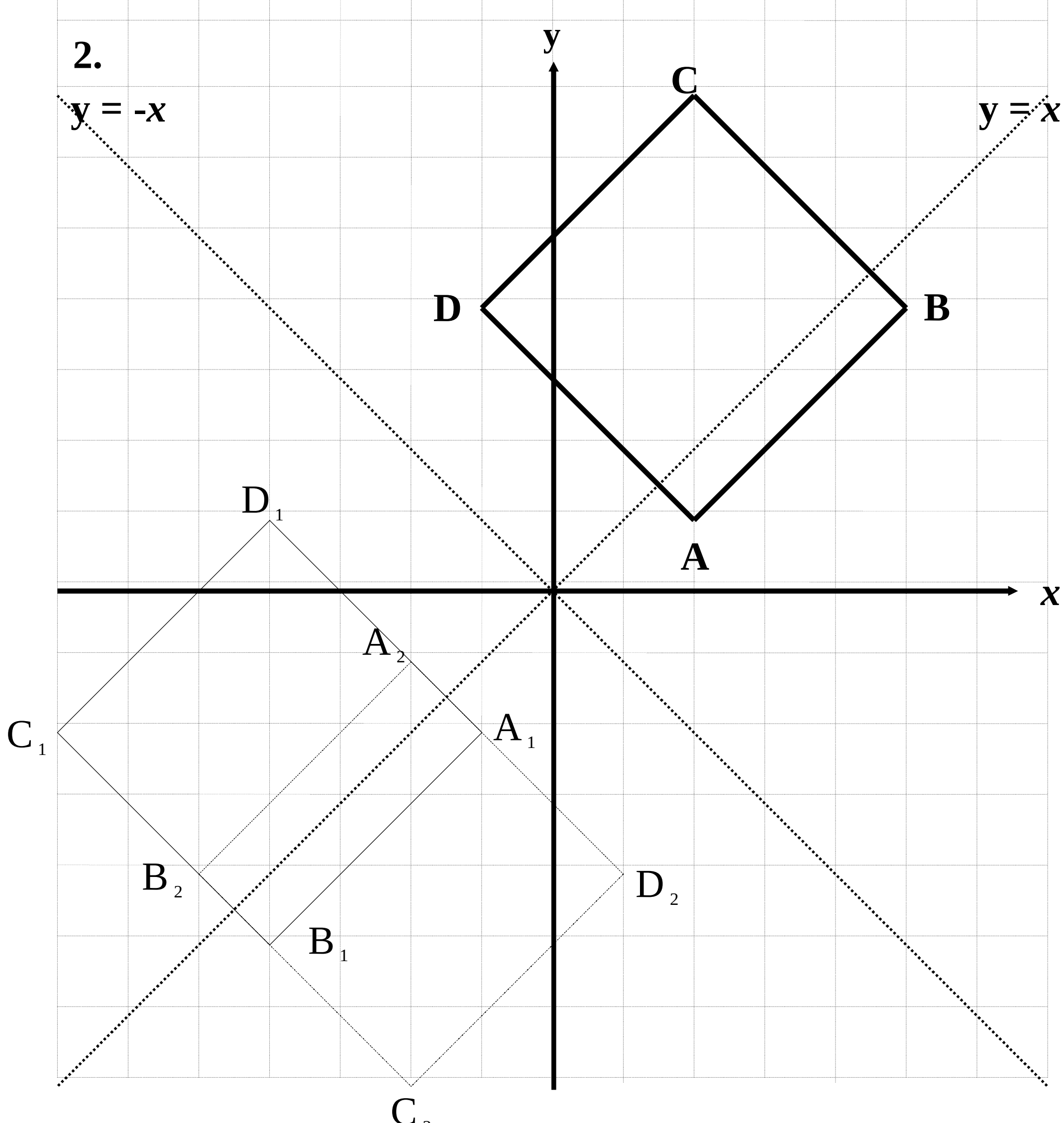
	$a - 3b = 1$ $-2a + b = 2$	$c - 3d = 3 \ldots \text{(i)}$ $-2c + d = 1 \ldots \text{(ii)}$
(i) × 2	$2a - 6b = 2$	$2c - 6d = 6 \ldots \text{(iii)}$
(ii) + (iii)	$-5b = 4$	$-5d = 7$

$$b = -\frac{4}{5} \qquad\qquad d = -\frac{7}{5}$$

$$a = -\frac{7}{5} \qquad\qquad c = -\frac{6}{5}$$

$$\therefore \quad T = \begin{pmatrix} -\frac{7}{5} & -\frac{4}{5} \\ -\frac{6}{5} & -\frac{7}{5} \end{pmatrix}$$


y (x = 0)
1.
B
A
C
C2
C1
x (y = 0)
A1
A2
B1
B2


2.
y
C
y = -x
y = x
D
B
D1
A
x
A2
C1
A1
B2
D2
B1
C2

SECTION 3

TRIGONOMETRY

Angles 0° to 360°, Graphs of Trigonometric Functions, Sine Rule, Cosine Rule and 3-Dimensional Problems

In GCSE Section 3, we studied trigonometry on right-angled triangles only.
In this section, the study is extended to cover angles between **0°** and **360°**, thereby making it possible to use trigonometry on ***all*** triangles.

1. ANGLES BETWEEN 0° AND 360°

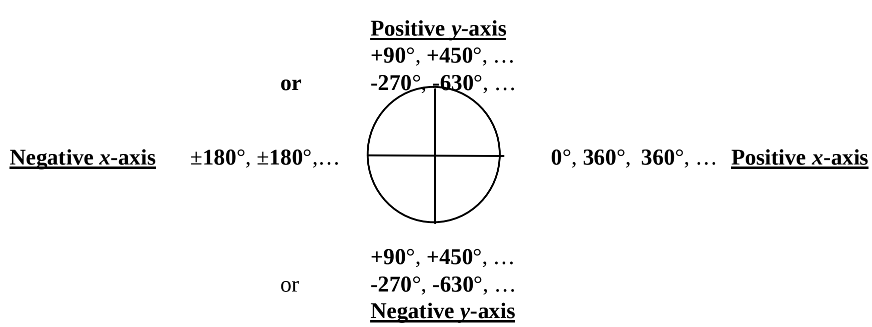
If we draw a **circle** and divide it into **four quadrants**, using a vertical line perpendicular to a horizontal line, both through the centre, we have a situation analogous to the Cartesian **coordinate axes**, meeting at the **origin (0,0)**.

It is important to remember that angles are **always** measured from the **positive side** of the ***x* – axis**, in an **anti-clockwise** direction if **positive**, and in a **clockwise** direction if **negative**. Imagine a **clock** with **one hand** which is **free to move** in **either** direction, but which **always starts** at **3 o'clock**. Clearly, then:

3 o'clock is equivalent to **0°**,

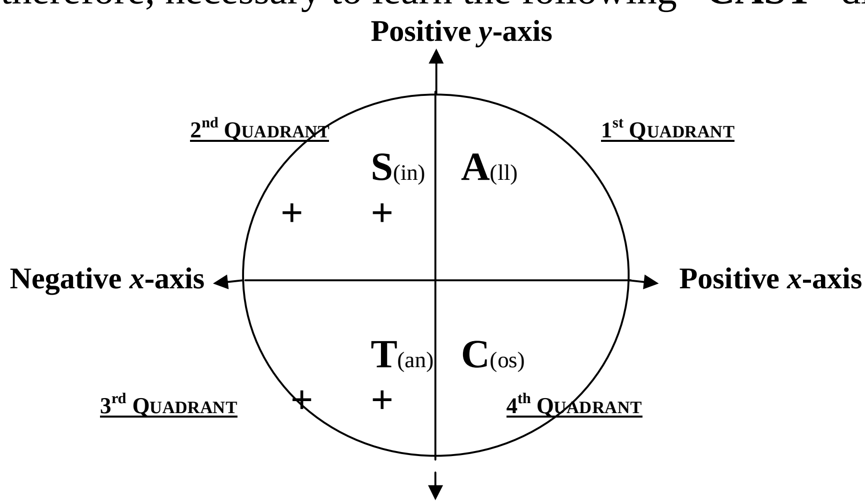
12 o'clock is equivalent to **+90°** or -270°,
depending on the **direction** of **movement** of the hand,

9 o'clock is equivalent to ±**180°**,
and **6 o'clock** is equivalent to **-90°** or **+270°**.



Angle	Sin(e)	Cos(ine)	Tan(gent)
0°	**0**	**1**	**0**
+90°	**1**	**0**	**+¥** (i.e. + infinity)
±180°	**0**	**-1**	**0**
+270°	**-1**	**0**	**- ¥** (i.e. -infinity)
±360°	**0**	**1**	**0**

When dealing with the **sin**, **cos** and **tan** of **any angle** between **0°** and **360°**, (other than those above), concentrate on the **acute angle** that the "**moving hand**" makes with the nearer **horizontal**, which is the **positive** or **negative *x*-axis**. This **acute angle** has the **same** trigonometric ratio (i.e. **sin**, **cos** or **tan**) as the **obtuse** or **reflex** angle being studied, but the ratio may be **positive** or **negative**.
It is, therefore, necessary to learn the following "**CAST**" diagram:

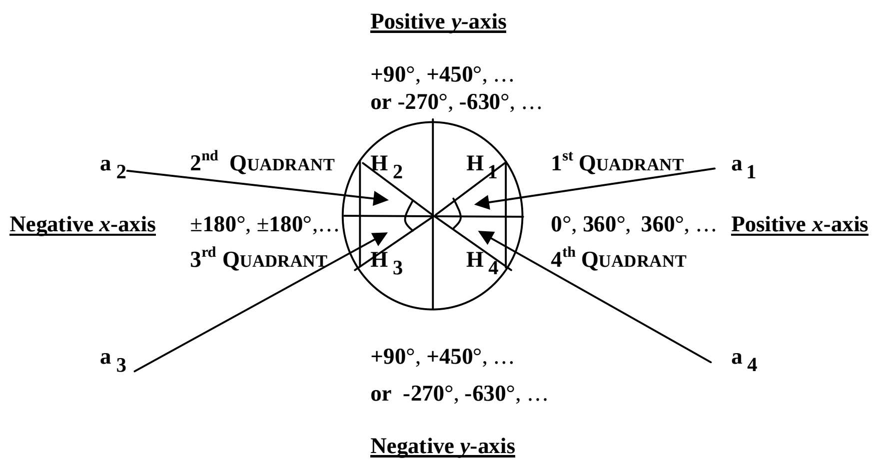


"CAST" DIAGRAM

Summary:

1st Quadrant:	**Sin**, **cos** and **tan** are **all positive**.
2nd Quadrant:	**Sin only** is **positive**.
3rd Quadrant:	**Tan only** is **positive**.
4th Quadrant:	**Cos only** is **positive**.

The normal **sign convention** used in drawing graphs applies and the **sign** of the "**moving hand**" is always **positive**.
In the diagram below, you can see that the "**moving hand**" always forms the **hypotenuse** ($\mathbf{H_1}$, $\mathbf{H_2}$, $\mathbf{H_3}$ and $\mathbf{H_4}$) of the **right-angled triangle**, containing the **acute angle** ($\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ and $\mathbf{a_4}$) which we are using, regardless of which one of the four quadrants it occupies.



"SIGN" DIAGRAM

In order to understand the **"CAST"** diagram, one needs to examine the **"SIGN"** diagram above, quadrant by quadrant:

(i) **<u>1st Quadrant</u>:**

$$\mathbf{Sin\ a_1} = \frac{+}{+} = +$$

$$\mathbf{Cos\ a_1} = \frac{+}{+} = +$$

$$\mathbf{Tan\ a_1} = \frac{+}{+} = +$$

Therefore the **sin**, **cos** and **tan** of **any angle** between **0°** and **90°** are **ALL positive**.

E.g. If we take a **1st quadrant** angle, **75°**, the acute angle, $\mathbf{a_1}$, is **75°** and we have:

	Sin 75°	=	**+0.966**	**(Correct to 3 decimal places)**
	Cos 75°	=	**+0.259**	**(Correct to 3 decimal places)**
and	**Tan 75°**	=	**+3.732**	**(Correct to 3 decimal places)**

(ii) 2^{nd} Quadrant:

$$\textbf{Sin } \mathbf{a_2} = \frac{+}{+} = +$$

$$\textbf{Cos } \mathbf{a_2} = \frac{-}{+} = -$$

$$\textbf{Tan } \mathbf{a_2} = \frac{+}{-} = -$$

Therefore the **sin only** of **any angle** between **90°** and **180°** is **positive**; **cos** and **tan** are **negative**.

E.g. If we take a **2^{nd} quadrant** angle, **105°** the acute angle, $\mathbf{a_2}$, is **75°** (i.e. 180° - 105°) and we have:

	Sin 105°	=	**+sin 75°**	=	**+0.966**
	Cos 105°	=	**-cos 75°**	=	**-0.259**
and	**Tan 105°**	=	**-tan 75°**	=	**-3.732**

(iii) 3^{rd} Quadrant:

$$\textbf{Sin } \mathbf{a_3} = \frac{-}{+} = -$$

$$\textbf{Cos } \mathbf{a_3} = \frac{-}{+} = -$$

$$\textbf{Tan } \mathbf{a_3} = \frac{-}{-} = +$$

Therefore the **tan only** of **any angle** between **180°** and **270°** is **positive**; **sin** and **cos** are **negative**.

E.g. If we take a **3^{rd} quadrant** angle, **255°**, the acute angle, $\mathbf{a_3}$, is **75°** (i.e. 255°- 180°) and we have:

	Sin 255°	=	**-sin 75°**	=	**-0.966**
	Cos 255°	=	**-cos 75°**	=	**-0.259**
and	**Tan 255°**	=	**+tan 75°**	=	**+3.732**

(iv) 4^{th} Quadrant:

$$\textbf{Sin } \mathbf{a_4} = \frac{-}{+} = -$$

$$\textbf{Cos } \mathbf{a_4} = \frac{+}{+} = +$$

$$\textbf{Tan } \mathbf{a_4} = \frac{-}{+} = -$$

Therefore the **cos only** of **any angle** between **270°** and **360°** is **positive**; **sin** and **tan** are **negative**.

E.g. If we take a **4^{th} quadrant** angle, **285°**, the acute angle, $\mathbf{a_4}$, is **75°** (i.e. 360°- 285°) and we have:

	Sin 285°	=	**-sin 75°**	=	**-0.966**
	Cos 285°	=	**+cos 75°**	=	**+0.259**
and	**Tan 285°**	=	**-tan 75°**	=	**-3.732**.

2. GRAPHS OF CIRCULAR FUNCTIONS

Sin, **cos** and **tan** functions are called **CIRCULAR** functions, since all points on their graphs lie on a unit circle (i.e. a circle of radius 1 unit), where the angle at the centre is measured in **RADIANS**.

Before proceeding with this study of their graphs, it is necessary to understand **radians**.

A **radian** may be described as the size of an **angle** at the **centre** of a **circle** which subtends an **arc equal** in length to the **radius**.

Since there are **2π radii** in the **full circumference**, clearly:

$$2\pi \text{ radians} = 360°$$

$$\therefore \quad \pi \text{ radians} = 180°$$

and $$\frac{\pi}{2} \text{ radians} = 90°, \text{ etc.}$$

To **change radians** to **degrees**, simply **multiply** by $\frac{180}{\pi}$

and to **change degrees** to **radians**, multiply by $\frac{\pi}{180}$.

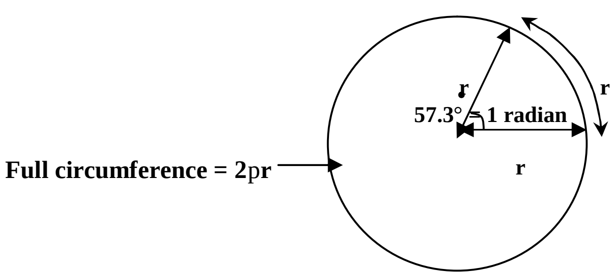
E.g.(i) Changing **1 radian** to **degrees**, we have:

$$1 \times \frac{180}{\pi} = 57.3°$$ (Correct to 1decimal place).

E.g.(ii) Changing **1°** to **radians**, we have:

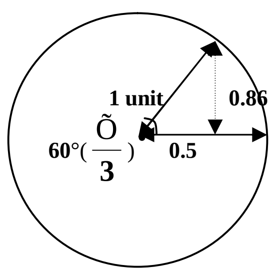
$$1 \times \frac{\pi}{180} = 0.017453292 \text{ radians}.$$

SEE DIAGRAM ON NEXT PAGE.



When drawing the graphs, the points are plotted, taking **angles** between **0** and **2**π **radians**, i.e. a **full revolution**.
Since this **range** of **angles** completes the **cycle** for **circular functions**, their **graphs** are **repetitions of the** "picture" between **0** and **2**π **radians**. It is this repetition which earns circular functions the name, **PERIODIC FUNCTIONS.** This means that
0 (degrees or radians), **360°** (**2**π **radians**), **720°** (**4**π **radians**), all have the same **sin**, **cos** and **tan**, as do **180°** (π **radians**),
540° (**3**π **radians**), … and **90°** ($\frac{\pi}{2}$ **radians**),
450° ($\frac{5\pi}{2}$ **radians**),… and so on. Therefore, **angles** of **any magnitude** or **sign** are "catered for", enabling us to draw their **graphs**.

However, before doing the **graphs** for the **sin**, **cos** and **tan** functions, it is helpful to one's understanding of this topic to draw a **unit circle** and **calculate** the sin, cos and tan of **any selected angle**, just by **taking measurements** from the diagram.
This is, in effect, using a scale drawing to "do one's own thing", and afterwards, the results can be compared with those provided by a scientific calculator (or mathematical tables).



N.B. Not drawn to scale.

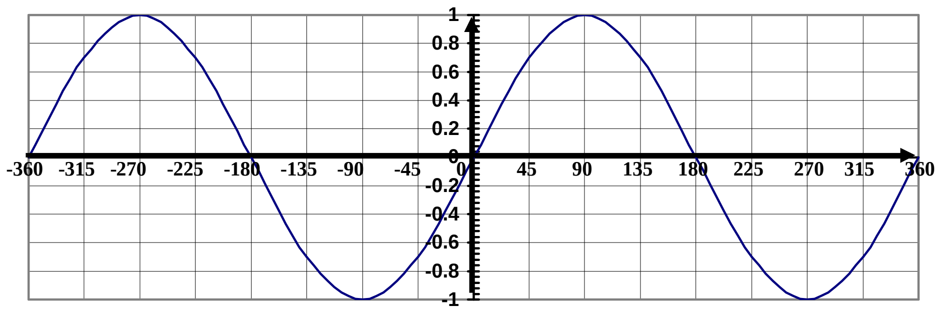
RESULTS	**SCALE DIAGRAM**	**SCIENTIFIC CALCULATOR**
Sin 60°($\frac{\pi}{3}$ radians)	**0.86**	**0.866**
Cos 60°($\frac{\pi}{3}$ radians)	**0.5**	**0.5**
Tan 60°($\frac{\pi}{3}$ radians)	**1.72**	**1.732**

Below is a table of values for **sin**, **cos** and **tan** for angles between **-360°** (**-2**p **radians**) and **+360°** (**+2**p **radians**).
(Notice that **sin** and **cos** both have **PERIOD 360°** and **tan** has **PERIOD 180°**):

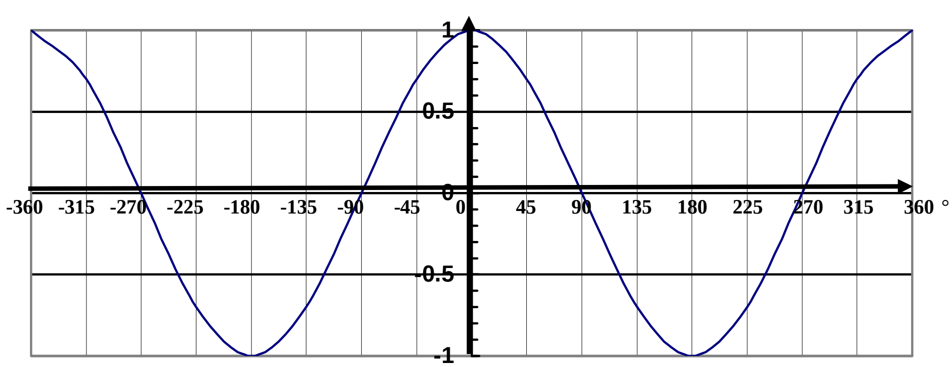
Angle ***x*°**	**-360**	**-315**	**-270**	**-225**	**-180**	**-135**	**-90**	**-45**	**0**
Angle ***x*** **radians**	-2Õ	$-\frac{7}{4}$Õ	$-\frac{3}{2}$Õ	$-\frac{5}{4}$Õ	-Õ	$-\frac{3}{4}$Õ	$-\frac{1}{2}$Õ	$-\frac{1}{4}$Õ	**0**
(i) **y = sin*x***	**0**	**0.7**	**1**	**0.7**	**0**	**-0.7**	**-1**	**-0.7**	**0**
(ii) **y = cos*x***	**1**	**0.7**	**0**	**-0.7**	**-1**	**-0.7**	**0**	**0.7**	**1**
(iii) **y = tan*x***	**0**	**1**	**¥**	**-1**	**0**	**1**	**¥**	**-1**	**0**

Angle ***x*°**	**+45**	**+90**	**+135**	**+180**	**+225**	**+270**	**+315**	**+360**
Angle ***x*** **radians**	$+\frac{1}{4}$Õ	$+\frac{1}{2}$Õ	$+\frac{3}{4}$Õ	+Õ	$+\frac{5}{4}$Õ	$+\frac{3}{2}$Õ	$+\frac{7}{4}$Õ	+2Õ
(i) **y = sin*x***	**0.7**	**1**	**0.7**	**0**	**-0.7**	**-1**	**-0.7**	**0**
(ii) **y = cos*x***	**0.7**	**0**	**-0.7**	**-1**	**-0.7**	**0**	**0.7**	**1**
(iii) **y = tan*x***	**1**	**¥**	**-1**	**0**	**1**	**¥**	**-1**	**0**

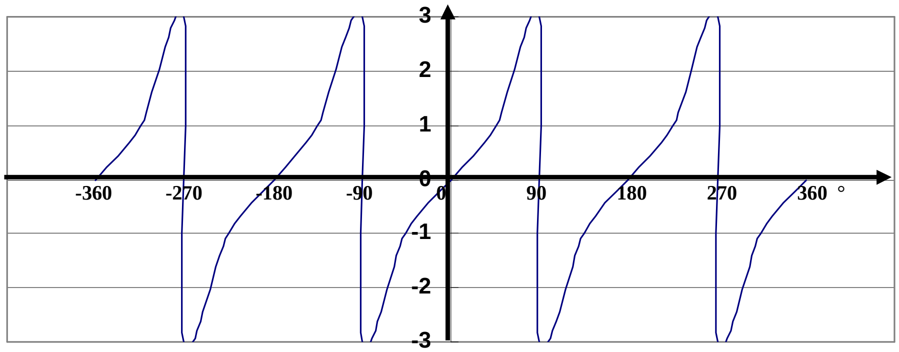
(i) *y* = sin *x* **(Period 360°)**



(ii) *y* = cos *x* **(Period 360°)**



(iii) $y = \tan x$ (Period 180°)

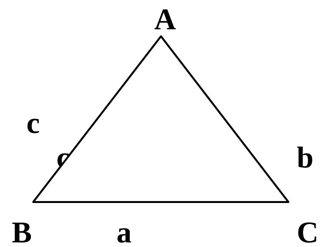


Notice how the graphs of **y = sin x** and **y = cos x** <u>oscillate</u> about the ***x*-axis** between ***y* = 1** and ***y* = -1**.

The graph of ***y*** is <u>piecewise</u>. Since ***y*** is **undefined** for ***x*** as any **odd multiple** of **90°**, it **tends towards infinity** at those points.

3. THE SINE RULE AND THE COSINE RULE

The **sine rule** and the **cosine rule** are used to solve **non-right-angled triangles**.



(i) The **Sine Rule** states:

$$\frac{\mathbf{a}}{\mathbf{SinA}} = \frac{\mathbf{b}}{\mathbf{SinB}} = \frac{\mathbf{c}}{\mathbf{SinC}}$$

NOTE: **Any one of these gives the length of the diameter of the circumscribing circle of Triangle ABC. Although knowledge of this is not required for G.C.S.E. mathematics, it is useful to know.**

When applying the **sine rule** to find a particular angle it is important to remember that the **sine** of an **angle** is **equal** to the **sine** of its **supplement**.

E.g. **30°** and **150°** both have the **same** sine, **0.5**.

Obviously, this can give rise to ambiguity in certain cases, so check that the correct size of angle is chosen:

E.g. If **sin A = 0.707**,
angle **A** may be **45°** or **135°** (i.e. **180° - 45°**).

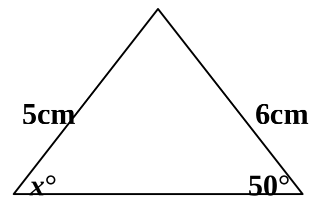
(ii) The **Cosine Rule** states:

$$\mathbf{a^2 = b^2 + c^2 - 2bc\cos A}$$
or $$\mathbf{b^2 = a^2 + c^2 - 2ac\cos B}$$
or $$\mathbf{c^2 = a^2 + b^2 - 2ab\cos C}.$$

TO DECIDE WHICH RULE IS APPLICABLE

(i) The **sine rule** may be used when the following information is given:

(a) **Two sides** and an **angle opposite** to **one** of them:



Using the sine rule to find ***x***°, we have:

$$\frac{5}{\sin 50°} = \frac{6}{\sin x°}$$

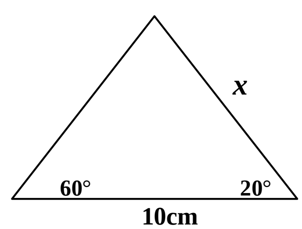
Þ **sin** $\boldsymbol{x}$**° = 0.9193**

\ $\boldsymbol{x}$**° = 66.8°**.

($\boldsymbol{x}$**° could normally be <u>180° - 66.8</u>°, but <u>not</u> <u>here</u> since the <u>sum</u> of the <u>three</u> <u>angles</u> in a triangle is <u>180</u>°).**

A **further application** of the **sine rule** would solve the triangle completely.

(b) **One side** and **any two angles**:



The **third** angle is **100°**.

Using the sine rule to find ***x***, we have:

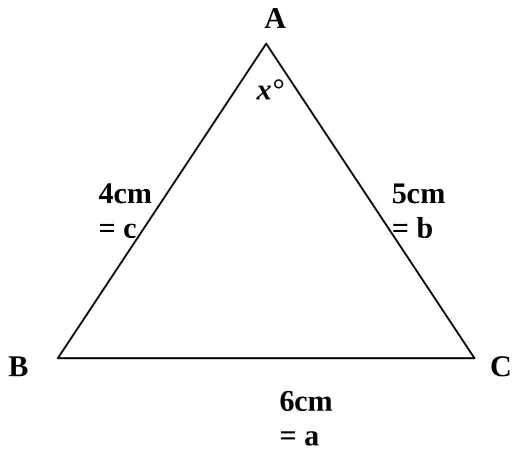
$$\frac{x}{\sin 60°} = \frac{10}{\sin 100°}$$

Þ $\boldsymbol{x}$ = **8.8cm**.

Again, a **further application** of the **sine rule** would solve the triangle completely.

(ii) The **cosine rule** must be used when the following information is given:

(a) Three sides only:

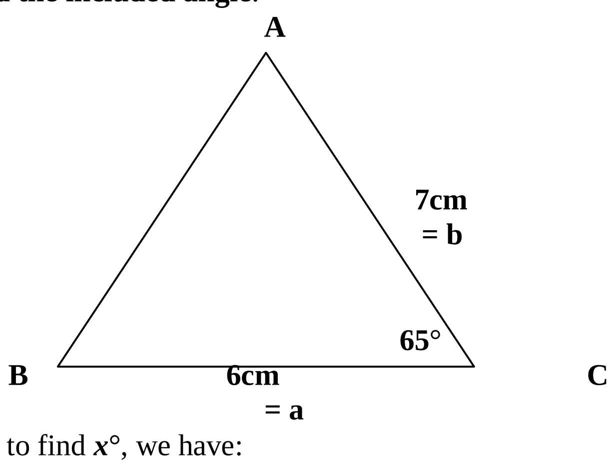


Using the **cosine rule** to find $x°$, we have:

$$\mathbf{a^2 = b^2 + c^2 - 2bc\cos A}$$
$$\Rightarrow \mathbf{36 = 25 + 16 - 40\cos A}$$
$$\Rightarrow \mathbf{\cos A = \frac{25 + 16 - 36}{40} = \frac{5}{40} = \frac{1}{8}}$$
$$\Rightarrow \mathbf{A = 82.8°}.$$

To find **another angle**, use the **sine rule** and the triangle is then solved completely.

(b) Two sides and the included angle:



Using the **cosine rule** to find $x°$, we have:

$$\mathbf{c^2 = a^2 + b^2 - 2ab\cos C}$$
$$\Rightarrow \mathbf{c^2 = 36 + 49 - 84\cos 65°}$$
$$\Rightarrow \mathbf{c^2 = 49.5}$$
$$\backslash \quad \mathbf{c = 7.04cm}.$$

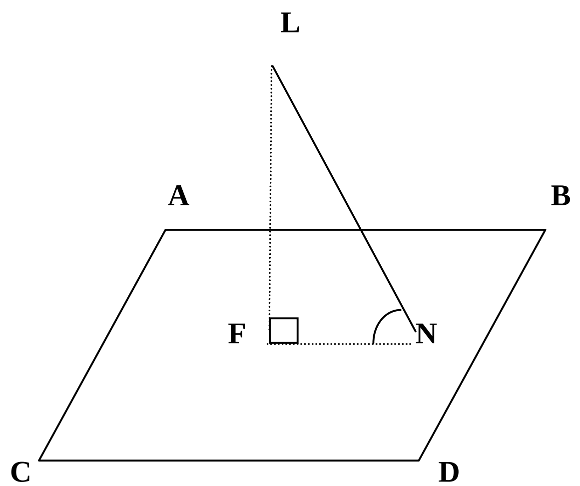
To find **another angle**, use the **sine rule** and the triangle is then solved completely.

4. THREE-DIMENSIONAL PROBLEMS

(i) To find the angle between a line and a plane:

Drop a **perpendicular** from a **point** on the **line** onto the **plane** and join the "**foot**" of this **perpendicular** to the **point** where the line **meets** the **plane**.

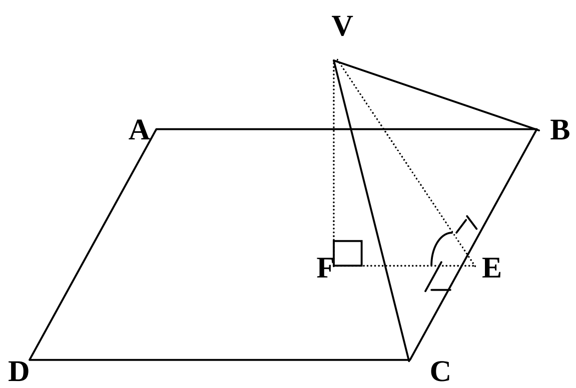
In the diagram below, the line **LN** meets the plane **ABCD** at **N**. The **angle** between **LN** and the plane **ABCD** is $\angle$ **LNF**.



(ii) To find the angle between two planes:

Draw a **perpendicular** on **each plane** onto the **common line** of intersection, to meet at a **point** on the common line.
The **angle** between these **two perpendiculars** is the **angle between** the **two planes**.

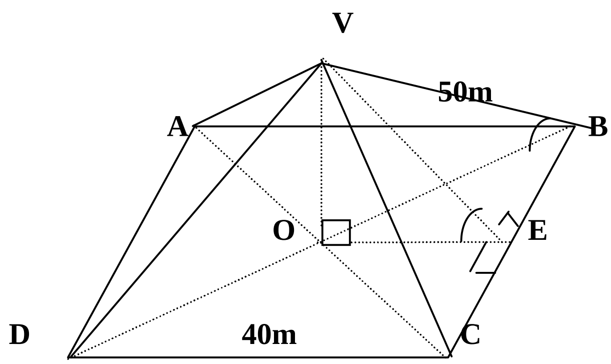
In the diagram below, the plane **VCB** meets the plane **ABCD** along the **common line BC**.
VE is drawn perpendicular to **BC** on the plane **VBC**
and **FE** is drawn perpendicular to **BC** on the plane **ABCD**.
The **angle** between the two planes is $\angle$ **VEF**.



WORKED EXAMPLE ON 3-DIMENSIONAL TRIGONOMETRY

The diagram below shows a **square pyramid**, base **ABCD** and vertex **V**, which is **vertically** above **O**. If the square base has **edge 40m** and **VB** is **50m**, find:

(i) the **angle** between the sloping edge **VB** and the base **ABCD**
and
(ii) the **angle** between the sloping face **VBC** and the base **ABCD**.

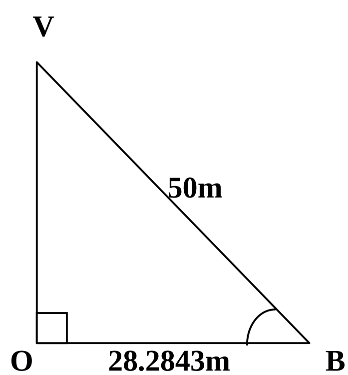


(i) The **angle** between **VB** and **ABCD** is Ð **VBO**.

Firstly, we must find the length of **OB**:

OB = $\frac{1}{2}\sqrt{40^2 + 40^2}$ = **28.2843m** **(by Pythagoras' Theorem)**.

Now, triangle **VOB** looks like this:



Using the **cos** ratio, we have:

Cos Ð VBO = $\frac{28.2843}{50}$

\ **Cos Ð VBO** = **0.565686**

Þ **Ð VBO** = **55.55°**.

(ii) If we draw a perpendicular from **V** onto **BC**, on plane **VBC**, **VE** bisects **BC**

(since VBC is an isosceles triangle),

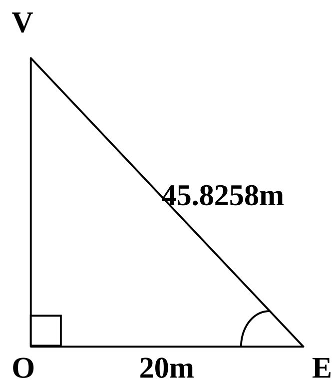
OE is perpendicular to **BC**, on plane **ABCD**

∴ Ð **VEO** is the **angle** between planes **VBC** and **ABCD**.

Firstly, we must find the length of **VE**, in triangle **VBC**:

VE = $\sqrt{50^2 - 20^2}$ = 45.8258m (by Pythagoras' Theorem).

Now, triangle **VOE** looks like this:



Using the **cos** ratio, we have:

Cos Ð VEO = $\frac{20}{45.8258}$

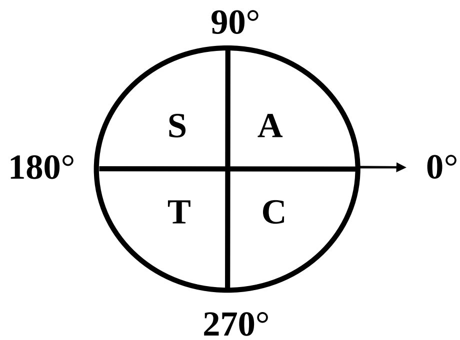
\ **Cos Ð VEO = 0.436435**

Þ **Ð VEO = 64.1°**.

EXERCISE 3

ANGLES OF 0° TO 360°, GRAPHS, SINE RULE, COSINE RULE & 3-D PROBLEMS

1.



With the aid of a **'CAST'** diagram, find:

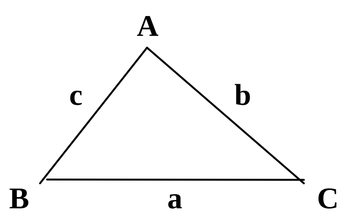
(a) **(i)** **cos 60°** **(ii)** **sin 120°** **(iii)** **tan 240°**
(iv) **cos 300°** **(v)** **tan 120°**

(b) the **2 values** of ***x*** between **0°** and **360°** where
(i) $\sin^{-1} x = 0.5$ **(ii)** $\cos^{-1} x = -0.5$ **(iii)** $\tan^{-1} x = 1$
(iv) $\sin^{-1} x = -0.5$ **(v)** $\cos^{-1} x = 0.5$ **(vi)** $\tan^{-1} x = -1$

2. Taking values of *x* from **-360°** (**-2p radians**) to **+360°** (**+2p radians**), draw the graphs of:-

(a) **y = sin *x*,**
(b) **y = cos *x***
and **(c)** **y = tan *x*.**

3.

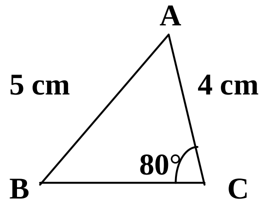


Sine Rule : $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

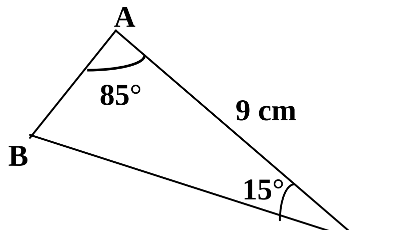
Cosine Rule : $a^2 = b^2 + c^2 - 2bc \cos A$
or $b^2 = a^2 + c^2 - 2ac \cos B$
or $c^2 = a^2 + b^2 - 2ab \cos C$.

Using the **Sine Rule** and / or the **Cosine Rule**, solve the following triangles **completely**:
(Answers to **1 decimal place**).

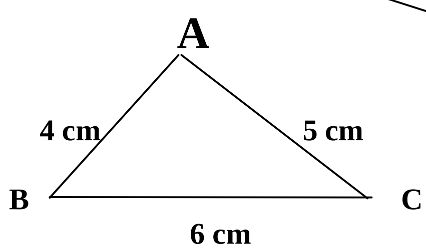
(a)



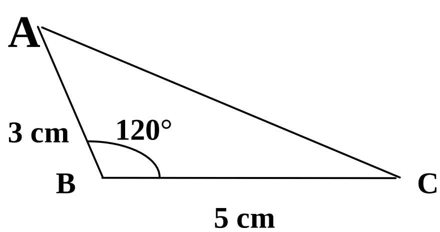
(b)



(c)

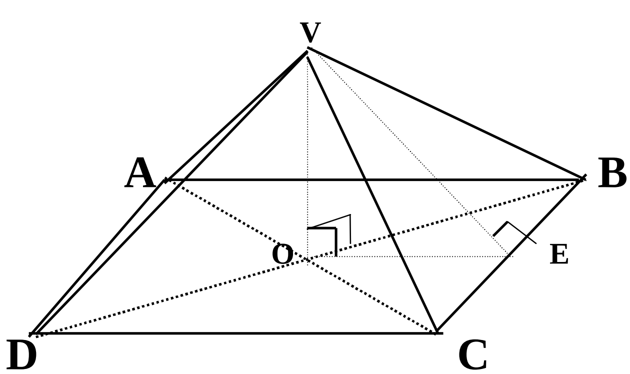


(d)



4. The diagram below shows a pyramid, base **ABCD** and vertex **V**, which is **vertically** above **O**. If the **square** base has edge **60 m** and **VO** is **40 m**, find :

(a) the **length** of the sloping edges **VA**, **VB**, **VC** and **VD**.
(b) the **angle** between the sloping edge **VB** and the base **ABCD**
and **(c)** the **angle** between the sloping face **VBC** and the base **ABCD**.



5. Using the following **identities** (which **you must learn** off by heart !) answer the questions below :

(i) $\sin^2 x + \cos^2 x = 1$ **(ii)** $\tan x = \dfrac{\sin x}{\cos x}$

(iii) $\text{sec(ant) } x = \dfrac{1}{\cos x}$ **(iv)** $\text{cosec(ant) } x = \dfrac{1}{\sin x}$

(v) $\text{cot(angent) } x = \dfrac{1}{\tan x}$ **(vi)** $\sec^2 x = 1 + \tan^2 x$

(vii) $\text{cosec}^2 x = 1 + \cot^2 x$ for **all** values of x.

(a) Rewrite the equation $3\sin^2 x = 3\cos^2 x - \cos x + 1$ in terms of $\cos x$.
Hence find **all** the **values** of x between **-180°** and **180°** which satisfy this equation.

(N.I.S.E.C. 1989)

(b) Show that the equation $6\cos x + \tan x = 5\sec x$ may be expressed in the form $6\sin^2 x - \sin x - 1 = 0$.
Find **all** the **values** of x between **-180°** and **+180°** which satisfy this equation.

(N.I.S.E.A.C. 1993)

(c)(i) By writing $\sec x$ and $\tan x$ in terms of $\sin x$ and / or $\cos x$, show that the equation $9\sec x + 5 = 12\tan x \sin x$ may be expressed in the form
$12\cos^2 x + 5\cos x - 3 = 0$.

(ii) ***Hence*** find **all values** of x between **-180°** and **180°** which satisfy this equation.
Give your answers to **two decimal places**.

(C.C.E.A. 1994)

GCSE (ADDITIONAL) EXERCISE 3 - ANSWERS

Angles of 0° to 360°, Graphs, Sine Rule, Cosine Rule & 3-D Problems

1. **(a)(i)** 0.5 **(ii)** 0.866 **(iii)** 1.732 **(iv)** 0.5 **(v)** -1.732

(b)(i) 30° and 150° **(ii)** 120° and 240° **(iii)** 45° and 225°

(iv) 210° and 330° **(v)** 60° and 300° **(vi)** 135° and 315°

2. **(a)**, **(b)** and **(c)**. See text.

3. **(a)** $\dfrac{5}{\sin 80^\circ} = \dfrac{4}{\sin B}$ Þ $\hat{B} = \mathrm{Sin}^{-1} \dfrac{4\sin 80^\circ}{5} = 52.0^\circ$

$$\hat{A} = 180^\circ - (80^\circ + 52^\circ) = 48^\circ$$

$\dfrac{a}{\sin 48} = \dfrac{5}{\sin 80}$ Þ $a = \dfrac{5\sin 48^\circ}{\sin 80} = 3.8$ **cm.**

(b) $\hat{B} = 180^\circ - (85^\circ + 15^\circ) = 80^\circ$

$\dfrac{9}{\sin 80} = \dfrac{a}{\sin 85}$ Þ $a = \dfrac{9\sin 85^\circ}{\sin 80} = 9.1$ **cm.**

$\dfrac{c}{\sin 15} = \dfrac{9}{\sin 80}$ Þ $c = \dfrac{9\sin 15}{\sin 80} = 2.4$ **cm.**

(c) $6^2 = 5^2 + 4^2 - 2\,(5)(4)\,\mathrm{Cos}\,A$

Þ $\hat{A} = \mathrm{Cos}^{-1}\ 0.125 = 82.8^\circ$

$\dfrac{5}{\sin B} = \dfrac{6}{\sin 82.8^\circ}$ Þ $B = \mathrm{Sin}^{-1} \dfrac{5\sin 82.8}{6} = 55.8^\circ$

Þ $C = 180^\circ - (82.8^\circ + 55.8^\circ) = 41.4^\circ$

(d) $b^2 = 5^2 + 3^2 - 2\,(5)(3)\,\mathrm{Cos}\,120^\circ$ Þ $b = 7$ **cm**

$\dfrac{5}{\sin A} = \dfrac{7}{\sin 120^\circ}$ Þ $A = \mathrm{Sin}^{-1} \dfrac{5\sin 120}{7} = 38.2^\circ$

$$\hat{C} = 180^\circ - (120^\circ + 38.2^\circ) = 21.8^\circ$$

4. **(a)** $OB = \dfrac{1}{2}\,DB = \dfrac{1}{2}\sqrt{60^2 + 60^2} = 42.4$ **m**

$VB = \sqrt{40^2 + 42.4^2} = 58.3$ **m**

\ **VA**, **VB**, **VC** and **VD** are each **58.3 m**.

(b) $V\hat{B}O = \mathrm{Tan}^{-1} \dfrac{40}{42.4} = 43.3^\circ$.

(c) $\quad OE = \frac{1}{2} DC = 30m.$

$\quad V\hat{E}O = \tan^{-1} \frac{40}{30} = 53.1°.$

5. (a) $3 \sin^2 x = 3 \cos^2 x - \cos + 1$

Þ $\quad 3(1 - \cos^2 x) = 3\cos^2 x - \cos x + 1$

Þ $\quad 3 - 3\cos^2 x = 3\cos^2 x - \cos x + 1$

Þ $\quad 6\cos^2 x - \cos x - 2 = 0$

Þ $\quad (3\cos x - 2)(2\cos x + 1) = 0$

Þ $\quad \cos x = \frac{2}{3} \text{ or } -\frac{1}{2}$

Þ $\quad x = \pm 48.2° \text{ or } \pm 120°.$

(b) $\quad 6\cos x + \tan x = 5\sec x$

Þ $\quad \frac{6\cos x}{1} + \frac{\sin x}{\cos x} = \frac{5}{\cos x}$

Þ $\quad \frac{6\cos^2 x + \sin x = 5}{\cancel{\cos} x}$

Þ $\quad 6(1 - \sin^2 x) + \sin x = 5$

Þ $\quad 6 - 6\sin^2 x + \sin x = 5$

Þ $\quad 6\sin^2 x - \sin x - 1 = 0$

Þ $\quad (3\sin x + 1)(2\sin x - 1) = 0$

Þ $\quad \sin x = -\frac{1}{3}$ Þ $x = -19.5° \text{ or } -160.5°$

$\quad \sin x = \frac{1}{2}$ Þ $x = 30° \text{ or } 150°$

(c) (i) $\quad 9\sec x + 5 = 12\tan x \sin x$

Q $\quad \sec x = \frac{1}{\cos x}$ and $\tan x = \frac{\sin x}{\cos x}$, we have:

$\quad \frac{9}{\cos x} + \frac{5}{1} = \frac{12\sin^2 x}{\cos x}$

Þ $\quad \frac{9 + 5\cos x = 12\sin^2 x}{\cancel{\cos} x}$

Q $\quad \sin^2 x = 1 - \cos^2 x$, we have

$\quad 9 + 5\cos x = 12(1 - \cos^2 x)$

Þ $\quad 9 + 5\cos x = 12 - 12\cos^2 x$

$\Rightarrow \quad 12\cos^2 x + 5\cos x - 3 = 0$

Q.E.D.

(ii) $\quad 12\cos^2 x + 5\cos x - 3 = 0$

Þ $\quad (3\cos x - 1)(4\cos x + 3) = 0$

Þ $\quad \cos x = \frac{1}{3} \text{ or } -\frac{3}{4}$

Þ $\quad x = \pm 70.5° \text{ or } \pm 138.6°$

SECTION 4

Statistics and Probability

Collecting, Ordering and Presenting Data

Raw Data:

Data which is **not organised numerically**.

E.g. The following temperatures were recorded over a 20-day period in the summer:
23°, 19°, 15°, 18°, 20°, 18°, 21°, 18°, 20°, 21°,
20°, 22°, 20°, 20°, 15°, 25°, 20°, 21°, 18°, 22°.

Organising Raw Data using **tally marks** to draw up a **frequency table**:

Score(°c)	Tally	Frequency	Score × Frequency
15	½½	**2**	**30**
18	½½½½	**4**	**72**
19	½	**1**	**19**
20	~~½½½½~~ ½	**6**	**120**
21	½½½	**3**	**63**
22	½½	**2**	**44**
23	½	**1**	**23**
25	½	**1**	**25**
Totals		20	396

Three types of Average (Measures of central location)**:**

(a) **Mean** **(Arithmetical Average)**
(b) **Median** **(**The **Middle** Score**)**
(c) **Mode** **(or** Modal **Score).**

(a) The mean is the ordinary **arithmetical average**
- **simply 'add them all up and divide by the number of them'.**

Here we have, as the **mean** daily temperature:

$$\frac{396}{20} = 19.8°C.$$

(b) The **median** is the **middle** score, when the scores are **ordered**, of an **odd** number of scores; when there is an **even** number of scores, the median is the **average** of the **two middle** scores.
Ordering the scores, we have:
15°, 15°, 18°, 18°, 18°, 18°, 19°, 20°, 20°, 20°,
20°, 20°, 20°, 21°, 21°, 21°, 22°, 22°, 23°, 25°.

Here, the **median** daily temperature is:

$$\frac{10th + 11th}{2} = \frac{20° + 20°}{2} = \mathbf{20°C}.$$

(c) The **mode** can be thought of as the **'fashionable'** score.
(*Mode* is French for 'fashion'.)
This means that it is the score with the **highest frequency**,

Clearly, then, the **modal daily temperature** is **20°C**.

Histogram:

It is customary to represent a **frequency distribution** by a **bar chart** or a **histogram**. A **histogram** is the display of data in the form of a block graph, where the **area** of each **rectangle** is **proportional** to the **frequency**.

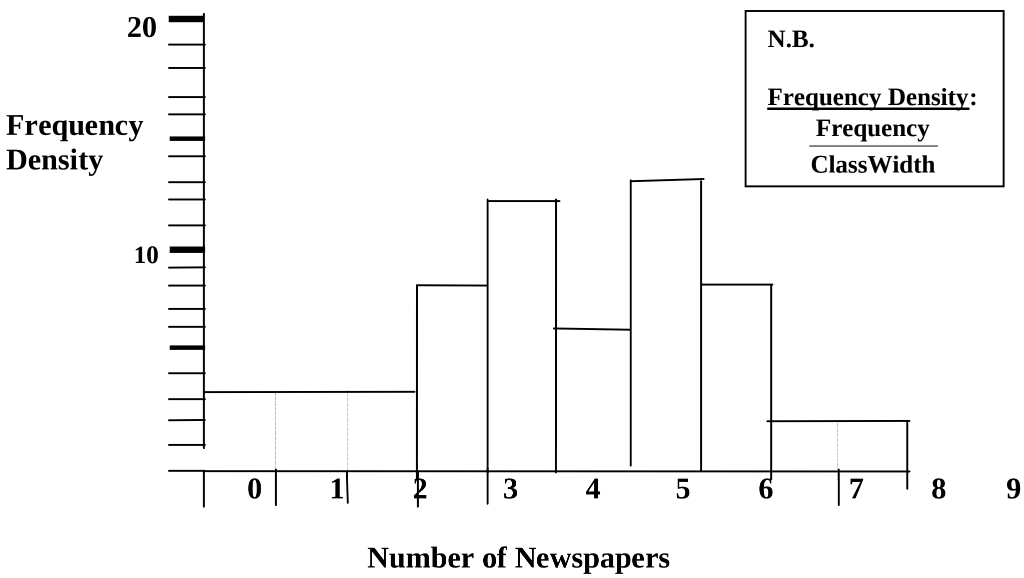
When the rectangles have the **same width**, their **heights** also are **proportional** to the **frequency**, making this type of histogram synonymous with a bar graph, in which the rectangles **always** have the same width.

E.g. A survey was conducted on **60** people to find out how many newspapers each person bought in a week.
The results are shown in the **frequency table** below.
Compile a **histogram** to represent the data.

Score (No. of Newspapers)	**0,1 or 2**	**3**	**4**	**5**	**6**	**7**	**8 or 9**
Frequency (No. of People)	**9**	**8**	**12**	**6**	**13**	**8**	**4**

(N.B. 9 must be divided by 3) **(N.B. 4 must be divided by 2)**

HISTOGRAM



GROUPED DISTRIBUTION

When a large amount of numerical data is being handled, it is convenient to group the scores into **classes**, and obtain a **class frequency** for each class.

The **table** on the next page gives a **grouped frequency distribution** for the **heights** (to the nearest cm) of **100** fifth-year pupils in a secondary school.

(**Note**: For later convenience, cumulative frequency, mid-points and f(x) are included in the table.)

Class Interval	Height (cm)	Frequency (f)	Cumulative Frequency	Mid-point of Class Interval (x)	f(x)
1st	145-149	5	5	147	735
2nd	150-154	8	13	152	1216
3rd	155-159	15	28	157	2355
4th	160-164	35	63	162	5670
5th	165-169	20	83	167	3340
6th	170-174	10	93	172	1720
7th	175-179	7	100	177	1239
	Totals	**100**			**16275**

Since these heights have been rounded off to the **nearest cm**, in theory each **class interval** contains heights of up to **0.5cm above** and **below** the **class limits**:

E.g. **Class 1**, **145-149** contains heights between **144.5cm** and **149.5cm**.
These figures give the **upper** and **lower class boundaries**.

It is important to note that each **class interval** has **width 5cm**, not **4cm**,
i.e. the **difference** between the **upper** and **lower class boundaries**:

149.5 - 144.5 = 5, **154.5 – 149.5 = 5**, …

This is particularly important when finding the **mid-point** of a **class interval**, which is required to find the **mean** of a distribution, or to draw a **histogram**, in which the **mid-points** of the **class intervals** must be at the **centres** of the **rectangles**.

To find the **mid-point** of a **class interval**, simply **add** the **upper** and **lower class boundaries** and **divide** by **2**.

THE MEAN OF A GROUPED DISTRIBUTION

To find the **mean height** of a pupil in the grouped frequency shown in the table above, **multiply** each class-interval's **mid-point** by its **frequency**, add these up and divide by 100, i.e. the **total number** of pupils.

We have:

$\sum f = 100$ and $\sum f(x) = 16275$

The **mean height** of a pupil is, therefore:

$$\frac{\sum f(x)}{\sum f} = \frac{16275}{100} = 162.75\text{cm}$$

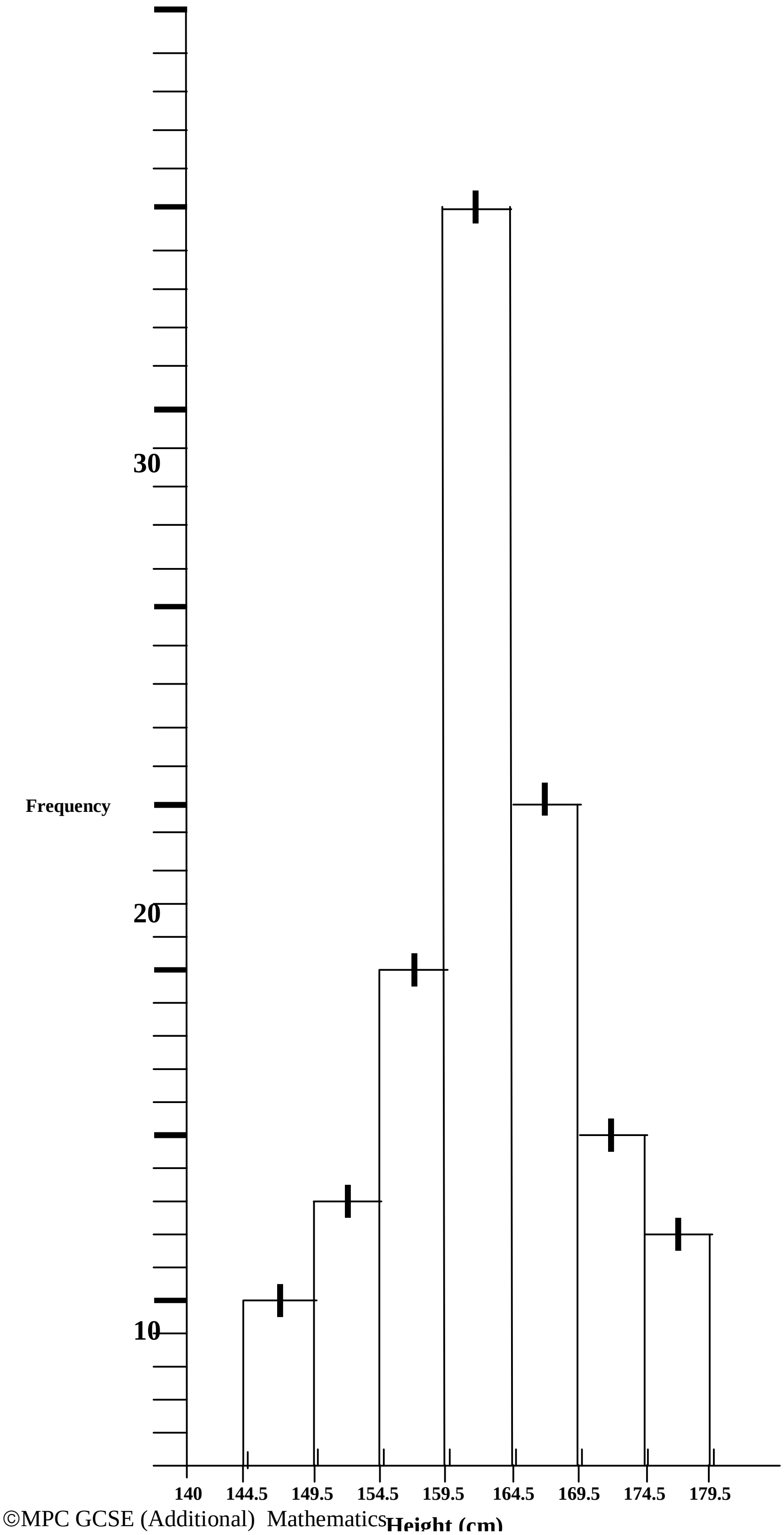
$$= 163\text{cm (to the nearest cm).}$$

HISTOGRAM FOR GROUPED DISTRIBUTION

A **histogram** for a **grouped distribution** must have the **mid-points** of the **class intervals** at the **centre** of each **rectangle**, representing the **frequency**. It must be emphasised that the **width** of each **rectangle** is the **difference** between **upper** and **lower class boundaries**, **NOT** the difference between **upper** and **lower class limits**.

Please examine the **histogram** below for the heights of the **100** pupils detailed in the table earlier. Remember that the **width** of each **rectangle** is **5cm**, not **4cm** (i.e. 149.5-144.5, 154.5-149.5,…).

Heights (to nearest cm) of 100 Fifth-Year Pupils



CUMULATIVE FREQUENCY DIAGRAM (OGIVE) FOR GROUPED DISTRIBUTION DETAILED IN THE TABLE EARLIER.

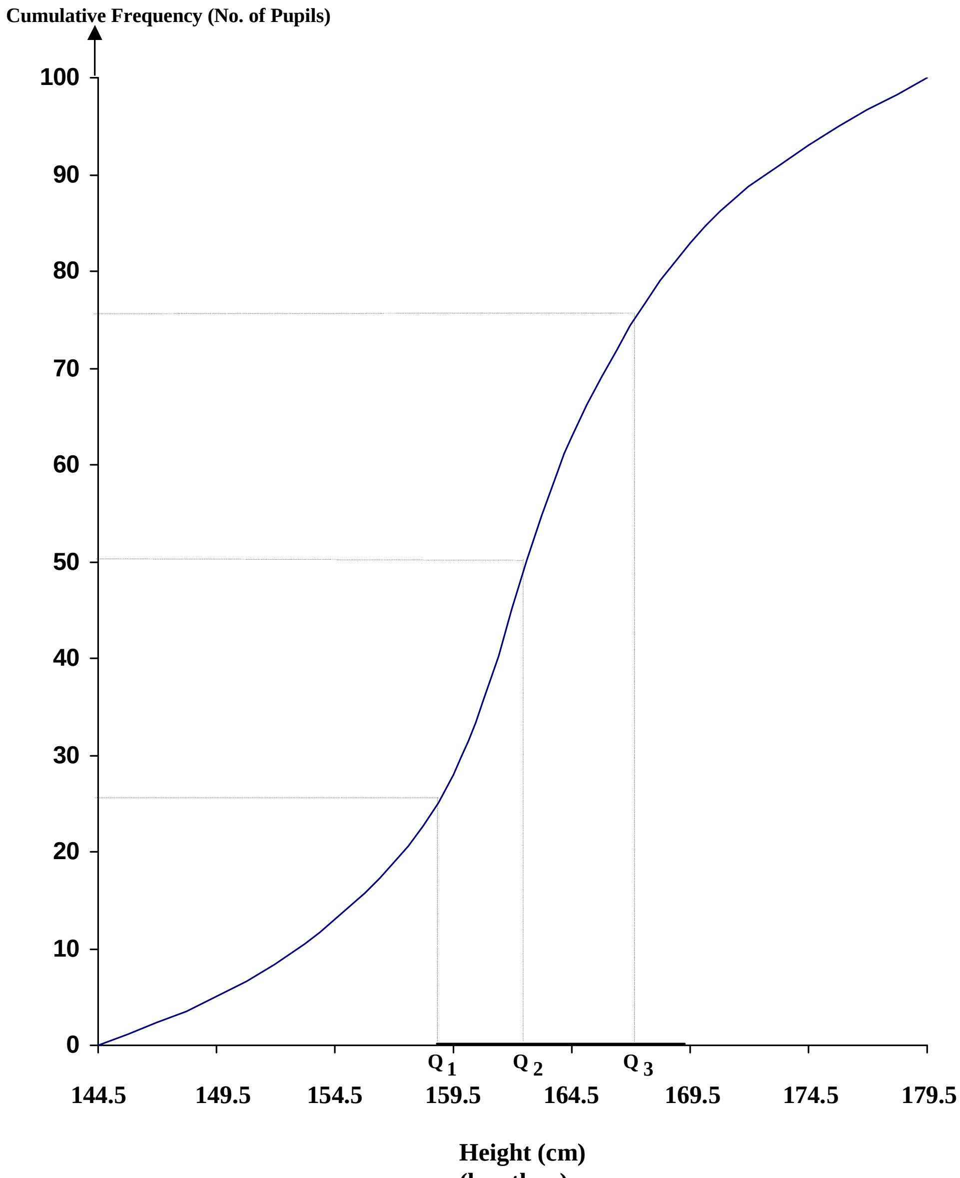
The main types of information that can be gained from the ogive are:

(a) the **interquartile range**
(b) the **median**.

If we divide the **total frequency** into **quarters**, the **score** corresponding to the **lower quarter** is the **lower quartile**, the **score** corresponding to the **middle** is the **median** and the **score** corresponding to the **upper quarter** is the **upper quartile**.
The **interquartile range** is the **difference** between the **score readings** provided by the **upper** and **lower quartiles**.

Now see below the **ogive** compiled from the information in the table.

OGIVE



From the cumulative frequency graph on the previous page, the readings are:

(a) **Interquartile Range**:

$$\mathbf{Q_3 - Q_1 = \quad 167.25 - 158.7 = \underline{8.55cm}}.$$

(b) **Median Height**:

$$\mathbf{Q_2 \quad = \quad \underline{162.75cm}}.$$

MEASURES OF DISPERSION OF DATA

Averages, being measures of **central location** only, do **not** indicate how the values in the data are **spread**.

The **range** in a set of data is the **difference** between the **highest value** and the **lowest value** in the set and this is the simplest measure of **dispersion** (or spread).

The **interquartile range** (dealt with above) indicates how the **middle half** of the set of data is spread and so gives a closer idea of the 'behaviour' of the bulk of the data.

As we have learnt earlier, $\mathbf{Q_1}$, $\mathbf{Q_2}$ and $\mathbf{Q_3}$ divide the data into quarters.

- If the difference between the **lower** quartile and the **median** is **less** than the difference between the **upper** quartile and the **median** the data values are said to have **positive skew**.

  This means: $\mathbf{Q_2 - Q_1 < Q_3 - Q_2 \Rightarrow}$ **positive skew**.

- If the difference between the **lower** quartile and the **median** is **more** than the difference between the **upper** quartile and the **median** the data values are said to have **negative skew**.

  This means: $\mathbf{Q_2 - Q_1 > Q_3 - Q_2 \Rightarrow}$ **negative skew**.

In our example considered earlier about heights of pupils, we have:

$$\begin{aligned} \mathbf{Q_2 - Q_1} &= \mathbf{162.75 - 158.7 = 4.05} \\ \mathbf{Q_3 - Q_2} &= \mathbf{167.25 - 162.75 = 4.5} \end{aligned}$$

Since $\mathbf{Q_2 - Q_1 < Q_3 - Q_2}$ this set of data has a slightly **positive skew**.

The **variance** of a set of data is the **mean** of the **squared deviations** from the **mean**.

- **Variance** $= \quad \dfrac{\sum (x_i - \bar{x})^2}{n} = \dfrac{\sum x_i^2}{n} - \bar{x}^2$.

The most commonly used measure of dispersion is the **standard deviation**.

The standard deviation is the **square root** of the **variance**.

- **Standard Deviation** $= \sqrt{\dfrac{\sum (x_i - \bar{x})^2}{n}}$ or $\sqrt{\dfrac{\sum x_i^2}{n} - \bar{x}^2}$.

Worked Example:

Calculate the **mean** and **standard deviation** of the **temperatures** recorded over the **20**-day period in the summer, detailed earlier.
Find also what **percentage** of the temperatures is within **1 standard deviation** of the mean.

N.B. We found the **mean** to be **19.8° C**.

Score x	Frequency f	Deviation from Mean x – 19.8	$(x - 19.8)^2$	$f(x - 19.8)^2$
15	2	-4.8	23.04	46.08
18	4	-1.8	3.24	12.96
19	1	-0.8	0.64	0.64
20	6	+0.2	0.04	0.24
21	3	+1.2	1.44	4.32
22	2	+2.2	4.84	9.68
23	1	+3.2	10.24	10.24
25	1	+5.2	27.04	27.04

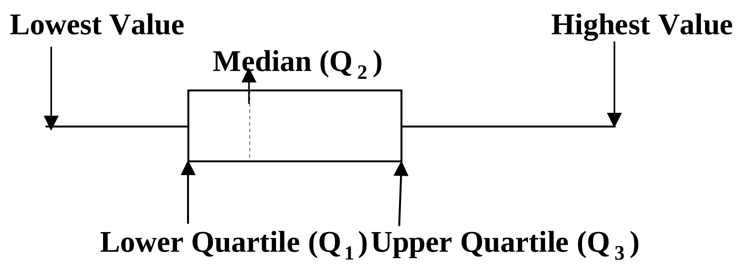
Sum of squared deviations: $\Sigma f(x - 19.8)^2 = 111.2$
Sum of frequency: $\Sigma f = 20$
Standard Deviation: $= \sqrt{\dfrac{\Sigma f(x - 19.8)^2}{\Sigma f}} = \sqrt{\dfrac{111.2}{20}} =$ **2.36°**.

Within the temperature range **19.8°** ± **2.36°**
i.e. **17.44°** to **22.16°**, we have **16** out of **20** i.e. **80%** of the scores.

DIAGRAMS

The use of diagrams often proves helpful when studying data and comparing distributions:

1. Box and whisker diagram



2. Stem and leaf diagram

Worked Example:

A class of twelve pupils were given tests in French and Spanish and the percentage marks are recorded in the table below.
Use a stem and leaf diagram to compare their performance in these subjects.

French	12	39	42	46	58	62	69	71	72	73	82	98	**Mean 60.3**
Spanish	39	41	46	51	53	54	63	67	69	78	79	81	**Mean 60.1**

French				**Test Marks (%)**			**Spanish**
				8	9		
				2	8	1	
	3	2	1	7	8	9	
	9	2	6	3	7	9	
	8	5	1	3	4		6
2	4	1	6				9
3	9						
				2	1		

N.B. The scores are ordered and recorded, in descending order, back to back from the central 'stem', reading backwards on the left and forwards on the right of the stem. Look at the third row where all the marks are seventy-something; the 71, 72 and 73 for French branch out from the central stem backwards and the 78 and 79 for Spanish branch out from the central stem forwards. (Separate diagrams, each with stem and right 'half-leaf', are often used.)
E.g. On the second-last row 9/3 reads 39 in French and 3/9 reads 39 in Spanish.

N.B. Whilst the **mean** marks for French and Spanish were about the **same** the **spread** of the marks in **French** is much **greater** than the spread of the marks in **Spanish**.

MOVING AVERAGES

Data reported at regular intervals of time tend to have an ***underlying trend***. This type of data can be 'smoothed out' using moving averages to form a ***trend line***, which can be extrapolated to predict future values.

Classifying Variations

(i) **Secular** trend is found when the direction of the data keeps going **upwards** or **downwards** over a long period of time:

E.g. 1.

The high jump record keeps **increasing** over a long period, giving an **upwards – moving secular trend** in the data.

E.g. 2

The winning time in a marathon keeps **decreasing** over a long time period, thereby giving a **downwards – moving secular trend** in the data.

(ii) **Seasonal** variation occurs when the data follow a pattern during corresponding months in successive years, for example heating bills. Electricity and heating bills fluctuate with the seasons - higher bills in the colder weather, lower bills in the warmer weather.

(iii) **Cyclical** variations occur when long periods of time follow the trend line, for example several years of prosperity in the economy followed by several years of recession, forming a pattern over time.

(iv) **Random** variations occur when unpredictable events like a war or a 'crash' in the stock markets happens. These variations **cannot** be 'smoothed out' using moving averages, since they are **irregular**.

A **trend line** can be constructed using moving averages

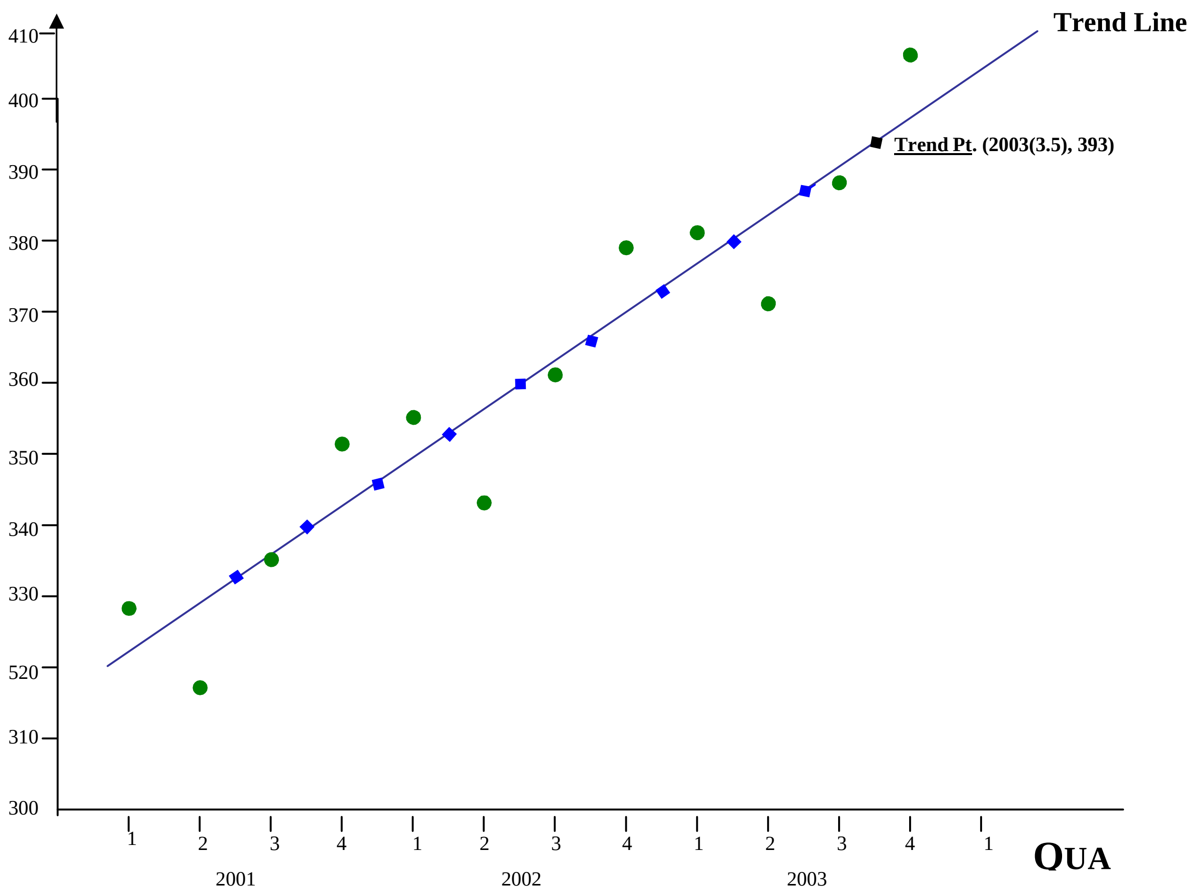
Example 1:

The table below shows the number of properties let by Nuhomes Estate Agency over the past three years.

	1st quarter	2nd quarter	3rd quarter	4th quarter
2001	328	317	335	351
2002	355	343	361	379
2003	381	371	388	406

(i) Plot these data on a graph.
(ii) Calculate appropriate moving averages to smooth the data.
(iv) Plot these averages on the graph and draw the trend line.
Showing clearly **where any reading is taken use the trend line to estimate how many properties are likely to be let in the first quarter of 2004.**
(v) Why do we use moving averages? (C.C.E.A. Additional paper 2 – 2004)

Method:
(i)



N.B. The **positioning** of the **moving average** is very important. **The moving average** must be plotted at the **mid-point** of the data from which it is calculated. E.g. The **1st moving average** is computed from quarters **1**, **2**, **3** and **4** of **2001**, giving the **1st moving average mid – way** between quarters **2** and **3** of **2001**, and so on . Using our trend line, the '**trend point'**, the **average** taken from quarters **2**, **3** and **4** of **2003** and quarter **1** of **2004**, gives **393 properties**. Working backwards, (**371 + 388 + 406 + x**) ÷ **4 = 393** gives **x = 407**, i.e. **407 properties estimated for 1st quarter of 2004**.

(ii) Graph in part (i).

4 – point moving averages

328		
317	332.75	328 + 317 +335 + 351 = 1331 ÷ 4 = 332.75
335	339.5	317 +335 + 351 + 355 = 1358 ÷ 4 = 339.5
351	346	335 + 351 + 355 + 343= 1384 ÷ 4 = 346
355	352.5	351 + 355 + 343 + 361 = 1410 ÷ 4 = 352.5
343	359.5	355 + 343 + 361 + 379 = 1438 ÷ 4 = 359.5
361	366	343 + 361 + 379 + 381 = 1464 ÷ 4 = 366
379	373	361 + 379 + 381 + 371 = 1492 ÷ 4 = 373
381	379.75	379 + 381 + 371 + 388 = 1519 ÷ 4 = 379.75
371	386.5	381 + 371 + 388 + 406 = 1546 ÷ 4 = 386.5
388		
406		

(iii) $$\frac{\mathbf{371 + 388 + 406 + x}}{\mathbf{4}} = \mathbf{393}$$

$$\mathbf{1165 + x = 1572}$$
$$\mathbf{x = 407}.$$

This means that our **estimate** of the number of properties likely to be let in the **1st quarter of 2004** is **407**.

(iv) We use moving averages to ‘smooth’ data so that we can try to predict future values.

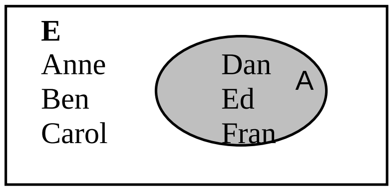
Probability

Set Theory - Venn Diagrams

Venn Diagrams (named after a 19th century mathematician called Joseph Venn) are used to **illustrate sets**.

The Universal Set **E** is represented by a **rectangle** and any set A of E is represented by a **closed curve** like a circle, **drawn within E**.

E.g.



Venn Diagram

The **Universal Set E** in the diagram above could represent a group of **6 children** and **A** could represent the **children** in the group who have **fair hair**.
We have:

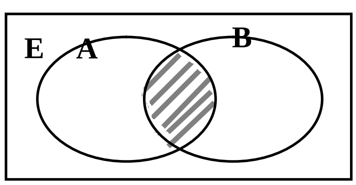
E = {Ann, Ben, Carol, Dan, Ed, Fran}.
A = {Dan, Ed, Fran}, i.e. fair –haired.
Then **A¢** (the **complement** of A, read *not* A) = {Ann, Ben, Carol},
i.e. **not** fair -haired.

N.B. **A + A¢ = E**.

Intersection

If we have 2 sets, A and B, within the Universal Set E, and some elements of **A** are *also* in **B**, then we have **intersection** of **A** and **B**, (i.e. overlap).

The Venn Diagram looks like this



The shaded region is A Ç B.

Diagram I (i)

The Universal Set E in the diagram above could be {1, 2, 3, 4, 5}.
A could be {even numbers less than 5};
B could be {prime numbers less than or equal to 5}.

Then E = {1, 2, 3, 4, 5},
A = {2, 4},
B = {2, 3, 5}.

So **A Ç B = {2}**.

- The symbol Ç reads ‘**and’** and corresponds to the ‘**And Law’** in probability which means that the **probabilities** are **multiplied**.

  The **Venn Diagram** would now look like this:

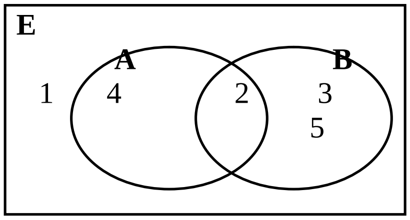


Diagram I (ii)

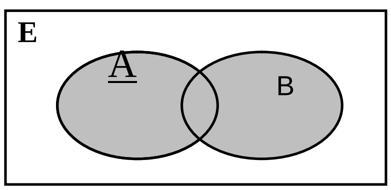
N.B. **A Ç B** = **{2}**.
(A Ç B)′ = **{1, 3, 4, 5}**.

1, 3, 4 and **5** are outside the intersection of A and B since they do not possess the property of being **even** and **prime**.

Union

If we have 2 sets A and B within the Universal Set E, the united elements of **A** and **B** give union.

The **Venn Diagram** looks like this:
(i)

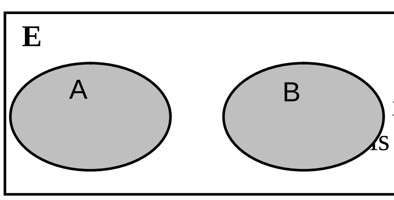


The shaded region is **A** È **B**.

Diagram U (i)

or

(ii)



Again, the shaded region is **A** È **B**

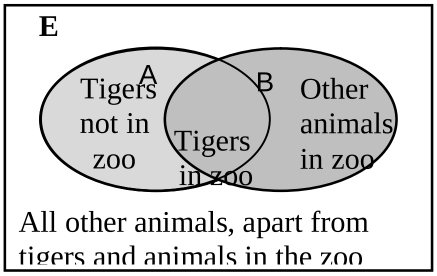
(**N.B.** Here that there is no intersection of A and B, but is union.)
Diagram U (ii)

In (i) above, E could be the set of all animals, A could be the set of all tigers and B could be the set of all animals in the zoo.

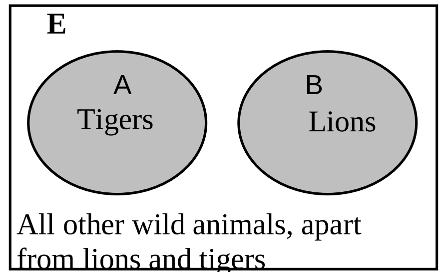
Then the **intersection** of **A** and **B** would be the set of all tigers in the zoo.
In (ii) above, E could be the set of all wild animals, A could be the set of all tigers and B could be the set of all lions. There is no intersection between A and B here since **no** animal is **both** a tiger and a lion; **A** and **B** are, therefore, **mutually exclusive sets**.

The **Venn Diagrams** would look like these:

(i) **Diagram U (iii)**



(ii) **Diagram U (iv)**



- The symbol È reads '**or**' and corresponds to the '**Or Law**' in probability which means that the **probabilities** are **added**.

N.B. (i) **A** È **B =** {All tigers plus all animals in the zoo *minus* all tigers in the zoo}.
(**A** È **B)′ =** {All animals *except* tigers and animals in the zoo}.

(ii) **A** È **B =** {All tigers *plus* all lions}.
(**A** È **B)′ =** {All wild animals *apart from* tigers and lions}.

Set theory and Venn Diagrams in particular, have useful applications in **probability** theory,

with the following interpretations:

- The universal set **E** is the **sample space**.
- The *elements* in the sets **A** and **B** are **outcomes**.
- The sets **A** and **B** themselves are **events**.
- **P(A) =** $\frac{n(A)}{N}$, where event **A** is a subset of all **N** equally likely outcomes.
- **P(A)** is the probability that event **A** occurs, **P(B)** is the probability that event **B** occurs, **P(AÇB)** is the probability that events **A** *and* **B** occur and **P(AÈB)** is the probability that events **A** *or* **B** or *both* events occur.
- **P(A′)** is the probability that event **A** does *not* occur, **P{(AÇB) ′}** is the probability that *both* **A** and **B** do *not* occur and so on.
- **P(A) + P(A′) = 1**, **P(AÇB) + P{(AÇB) ′} = 1**, and so on, i.e. the probability of an event taking place *added* to the probability of that event *not* taking place is **1**.
  N.B. **A** and **A′** are termed **mutually exclusive** events.

Probability – ‘And’ and ‘Or’ Laws

- $\cap$ in **set theory** corresponds to ‘**and’**, i.e. ‘**multiplication**’ in **probability theory**.

  In **Diagram I (i)**, we have the probability of event **A** *and* event **B** as:

$$\mathbf{P(A \cap B)} = \frac{1}{5}.$$

  Conditional probability means the probability that event **A** occurs, given that event **B** *has occurred already*; the **notation** used for this event is **P(A/B).**
  In this case the multiplication law gives:

$$\mathbf{P(A \cap B)} = \mathbf{P(A/B) \times P(B)}$$

$$\Rightarrow \quad \mathbf{P(A/B)} = \frac{\mathbf{P(A \cap B)}}{\mathbf{P(B)}} \quad \text{(By rearrangement of the above.)}$$

  The **events** are said to be **independent** when the probability of either event occurring is *unaffected* by the probability of the *other* event having occurred.
  In this case the multiplication law gives:

$$\mathbf{P(A \cap B)} = \mathbf{P(A) \times P(B)}.$$

- $\cup$ in **set theory** corresponds to ‘**or’**, i.e. ‘**addition**’ in **probability theory**.

  In **Diagram U (i)**, we have the probability of event **A** *or* event **B** as:

$$\mathbf{P(A \cup B)} = \mathbf{P(A) + P(B) - P(A \cap B)}.$$

  In **Diagram U (ii)**, **A** and **B** are **independent events**, so the probability of event **A** *or* event **B** is: $\mathbf{P(A \cup B)} = \mathbf{P(A) + P(B)}$.

  N.B. If **A** and **B** are **mutually exclusive** events, $\mathbf{P(A \cap B) = 0}$ and $\mathbf{P(A \cup B) = P(A) + P(B)}$.

Worked Example:

A and **B** are independent events.
Given that **P(A) = 0.2** and **P(B) = 0.4**, find:

(i) $\mathbf{P(A \cap B')}$.
(ii) $\mathbf{P(A \cup B)}$.

Method:

(i) $\because$ **A** and **B** are independent, **A** and **B′** are independent

$$\Rightarrow \mathbf{P(A \cap B')} = \mathbf{P(A) \times P(B')} = \mathbf{0.2 \times 0.6} = \mathbf{0.12}.$$

(ii) $$\mathbf{P(A \cup B)} = \mathbf{P(A) + P(B) - P(A \cap B)}.$$

$\because$ **A** and **B** are independent, $\mathbf{P(A \cap B) = P(A) \times P(B)}$

$$\Rightarrow \mathbf{P(A \cup B)} = \mathbf{0.2 + 0.4 - 0.2 \times 0.4} = \mathbf{0.52}.$$

TREE DIAGRAM

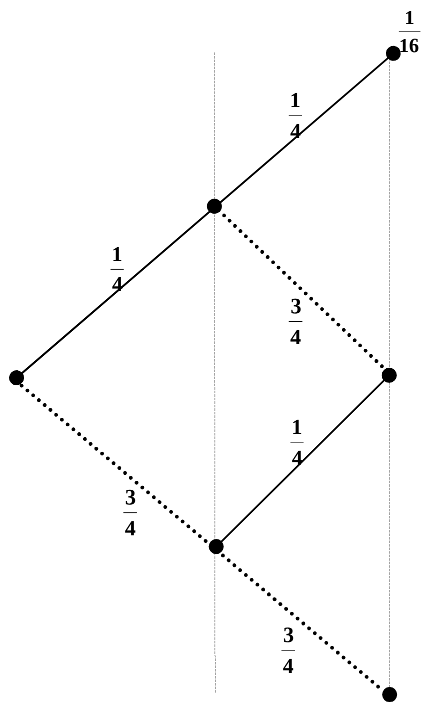
A **tree diagram** may be drawn to represent probabilities when only **two possible outcomes** have to be considered, for example tossing a coin, where the outcome at each toss can only be **one** of **two** possibilities, Heads or Tails.

Example:
A **biased** coin is tossed **twice**. The probability of obtaining **2 Heads** is $\frac{1}{16}$.

Use a tree diagram to answer the following questions:

(i) Find the probability of obtaining **2 Tails**.
(ii) Find the probability of obtaining a **Head** and a **Tail**.

First Toss **Second Toss**



Method:

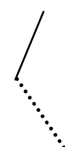
The probability of two Heads means:

P(1st Head) and P(2nd Head)

$\Rightarrow$ **P(Head) $\times$ P(Head)** $= \frac{1}{16}$

$\therefore$ **P(Head)** $= \sqrt{\frac{1}{16}} = \frac{1}{4}$.

[The tree may be continued with more ‘branches’, as required:



If the probability of obtaining 3 Heads was given as say $\frac{1}{27}$, then $\sqrt[3]{\frac{1}{27}} = \frac{1}{3}$ is the probability of a Head for each toss; in this case the probability of a Tail is $\frac{2}{3}$.]

If the probability of a **Head** is $\frac{1}{4}$, the probability of a **Tail** is $\frac{3}{4}$.
(The two together must add up to **1**).

Answer:

(i) **P(2 Tails)** $=$ **P(1st Tail) <u>and</u> P(2nd Tail)**

$$= \frac{3}{4} \times \frac{3}{4}$$

$$= \frac{9}{16}.$$

(ii) **P(1 Head and 1 Tail)**

$=$ **P(1st Head) and P(2nd Tail) or P(1st Tail) and P(2nd Head)**

$$\Rightarrow \left(\frac{1}{4} \times \frac{3}{4}\right) + \left(\frac{3}{4} \times \frac{1}{4}\right)$$

$$= \frac{3}{16} + \frac{3}{16}$$

$$= \frac{3}{8}.$$

CORRELATION

A set of **paired** observations from **two** random variables is called a ***bivariate distribution***. If, in a bivariate distribution, a **change** in **one** variable is **matched** by a **similar proportional change** in the **other** variable, then the technique used to measure the degree of association is called **correlation**.
Before quantifying the interdependence of two variables it is often useful to plot the paired observations on a **scatter diagram**. By inspection, if a straight line can be drawn to fit the data reasonably well, then there exists a **linear correlation** between the variables:

Positive correlation: *both* variables **increase** together.
Negative correlation: as *one* variable **increases**, the *other* **decreases**.

Rank Correlation - Spearman's Coefficient

If the *actual values* of the data are *not* given, but the ranks in which the data are ordered *are* given, a correlation coefficient based on the *ranks* can be determined.
The most popular of the methods used for determining a coefficient for rank correlation is **Spearman's coefficient of rank correlation**.

Spearman's Coefficient of Rank Correlation:

$r_s \quad = \quad 1 - \dfrac{6\sum d_i^2}{n^3 - n}$, where $\mathbf{d = x_i - y_i}$, the **difference** in the values of the **ranks** between pairs.

Worked Example 2:

Calculate **Spearman's rank correlation coefficient** for the following data:

x	**25**	**27**	**27**	**28**	**29**	**31**	**32**	**33**	**34**	**34**
y	**45**	**49**	**51**	**54**	**52**	**60**	**60**	**62**	**63**	**64**

Method:

The sets of values for **x** and **y** must first be ranked. Where two or more *equal* values occur, the rank assigned to **each** is the **average** of the **positions occupied** by the **tied values**. Then we have:

x	rank	y	rank	point (x, y)	difference in ranks (d)	d^2
34	$1\frac{1}{2}$	64	1	(25, 45)	0	0
34	$1\frac{1}{2}$	63	2	(27, 49)	$\frac{1}{2}$	$\frac{1}{4}$
33	3	62	3	(27, 51)	$-\frac{1}{2}$	$\frac{1}{4}$
32	4	60	$4\frac{1}{2}$	(28, 54)	-1	1
31	5	60	$4\frac{1}{2}$	(29, 52)	1	1
29	6	54	6	(31, 60)	$-\frac{1}{2}$	$\frac{1}{4}$
28	7	52	7	(32, 60)	$\frac{1}{2}$	$\frac{1}{4}$
27	$8\frac{1}{2}$	51	8	(33, 62)	0	0
27	$8\frac{1}{2}$	49	9	(34, 63)	$\frac{1}{2}$	$\frac{1}{4}$
25	10	45	10	(34, 64)	$-\frac{1}{2}$	$\frac{1}{4}$

$$\sum d^2 = 3\tfrac{1}{2}$$

$$r_s = 1 - \frac{6\sum d_i^2}{n^3 - n} \quad \Rightarrow \quad r_s = 1 - \frac{6 \times 3.5}{1000 - 10} = \frac{323}{330}.$$

N.B. **-1** represents a perfect **negative** correlation and **+1** a perfect **positive** correlation.)

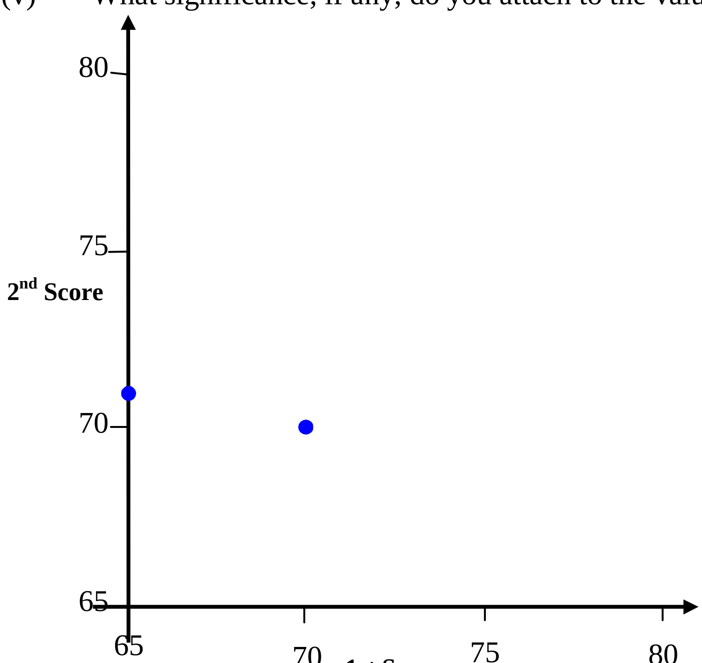
Worked Example - Correlation

<u>At a recent talent contest competitors were scored on each of 2 performances.</u>
<u>The table below shows their scores for each performance.</u>

Competitor	Kate	Paul	Callum	Danielle	Ewan	Hannah
Performance 1	**71**	**68**	**77**	**65**	**70**	**75**
Performance 1	**74**	**73**	**76**	**71**	**70**	**72**

<u>These data are plotted on the graph given in the figure below.</u>

(i) Draw the line of best fit on this graph.
(ii) Determine the equation of the line of best fit which you have drawn.
(iii) Work out the rank orders for the scores in each performance.
(iv) Calculate Spearman's coefficient of rank correlation.
(v) What significance, if any, do you attach to the value you obtained in (iv)?

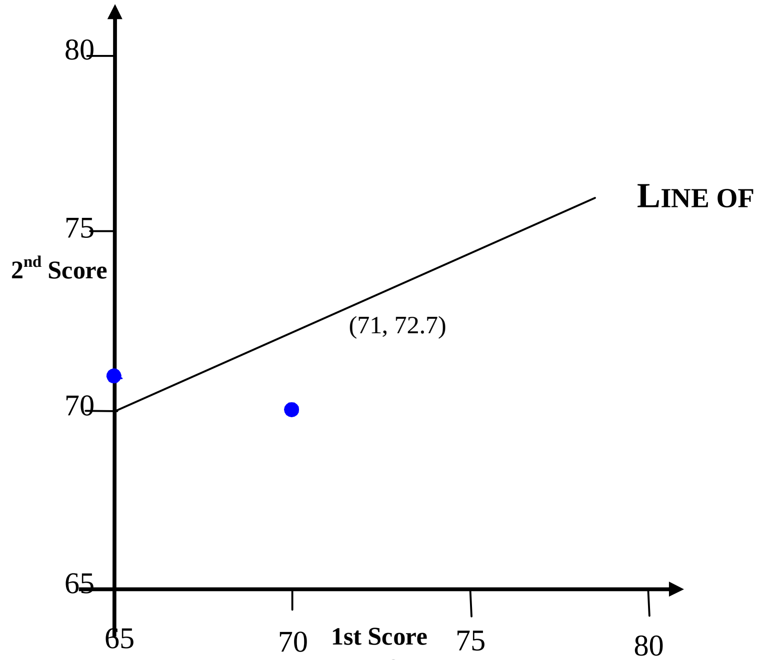


CCEA Paper 2, 2004)

(i) **Mean** – Performance **1**: (71 + 68 + 77 + 65 + 70 + 75) ÷ 6 = **71**.

Mean – Performance **2**: (74 + 73 + 76 + 71 + 70 + 72) ÷ 6 = $\mathbf{72\frac{2}{3}}$.

N.B. (**71**, $\mathbf{72\frac{2}{3}}$) must be plotted on the graph and the line of best fit must go through this point.



(ii) $m = \dfrac{2\frac{2}{3}}{6} = \dfrac{8}{18} = \dfrac{4}{9}$.

$(x_1, y_1) = (71, 72\frac{2}{3})$.

$y - y_1 = m(x - x_1)$

$\Rightarrow \quad y - 72\frac{2}{3} = \frac{4}{9}(x - 71)$

(× 9): $\Rightarrow \quad 9y - 654 = 4x - 284$

$\Rightarrow \quad 9y = 4x + 370$ (or any rearrangement of this.)

(iii)

x	rank	y	rank	point (x, y)	difference in ranks (d)	d^2
77	1	76	1	(71, 74)	2 – 3 = -1	1
75	2	74	2	(68, 73)	3 – 5 = -2	4
71	3	73	3	(77, 76)	1 – 1 = 0	0
70	4	72	4	(65, 71)	5 – 6 = -1	1
68	5	71	5	(70, 70)	6 – 4 = 2	4
65	6	70	6	(75, 72)	4 – 2 = 2	4
					$\sum d^2 =$	14

(iv) $r_s = 1 - \dfrac{6\sum d_i^2}{n^3 - n} \quad \Rightarrow \quad r_s = 1 - \dfrac{6 \times 14}{6^3 - 6} = 1 - \dfrac{84}{210} = 0.6.$

(v) There is a slight positive correlation.

NOTE: A *large* value of the correlation coefficient, **r** indicates a *strong* correlation, but this does not necessarily mean that there is a causal relationship between the two variables.

GCSE(ADDITIONAL) - EXERCISE 3

Part 1 - Statistics

1. A survey was conducted on **60** children to find out how many times they visit the doctor in a year. The results are shown in the table below.

Compile a **histogram** to represent these data.

Score (No. of visits)	**0, 1** or **2**	**3**	**4**	**5**	**6**	**7**	**8** or **9**
Frequency (No. of children)	**12**	**6**	**10**	**11**	**8**	**7**	**6** (N.B. 6 must be divided by 2)

(N.B. 12 must be divided by 3)

2. In a survey, **120** motorists were requested to record the **petrol consumption** of their cars in **kilometres** per li**t**re, rounded off to the nearest **km/l**.

Below is the frequency distribution obtained from the set of data:

Km/l	**9-10**	**11-12**	**13-14**	**15-16**	**17-18**	**19-20**	**21-22**
No. of **Cars**	**5**	**7**	**21**	**41**	**24**	**15**	**7**

(a)(i) State the **modal class** of this distribution.
(ii) Calculate the **mean** petrol consumption of a car.
(**N**.**B**. Use **mid-points** of class intervals).

(b)(i) Draw a **histogram** to represent the above data.
(**N**.**B**. **Width** of each rectangle is **2 km/l** - 8.5 - 10.5, 10.5 - 12.5, etc.)

(ii) On the histogram for **(i)**, superimpose a **frequency polygon**.

(c)(i) Complete a **'less than' cumulative frequency** table:

km/l	**8.5**	**10.5**	**12.5**	**14.5**	**16.5**	**18.5**	**20.5**	**22.5**
Frequency	**0**	**5**	. . .	. . .	. . .	. . .	. . .	. . .

(ii) On graph paper, draw the **cumulative frequency curve** (ogive) from your completed cumulative frequency table at **(i)**.

By drawing broken lines on your graph, answer the questions below.

(iii) Estimate
(1) the **median**;

(2) the **inter-quartile range** of the distribution.

(iv) An 'average' petrol consumption is one which lies between **13.5** and **18.5 kilometres per litre**.
Estimate, from your ogive, the number of cars where petrol consumption was 'average'.

(v)(1) Those cars, whose recorded petrol consumption was **less than 12 km/l**, were selected to take part in a rally.
What **percentage** of the cars were eligible for the rally?

(2) If the **probability** of the winning car in the rally being an **MG 'Midget'** is **0.4**, how many MG 'Midgets' took part in the rally?

3. In a survey, **20** young people were asked to record the number of hours they spent on exercise each week. The results were as follows:

15	**10**	**7**	**10**	**9**	**6**	**9**	**7**	**10**	**9**
12	**9**	**10**	**10**	**5**	**9**	**7**	**7**	**10**	**0**

Calculate the **mean** and **standard** deviation for the above data.

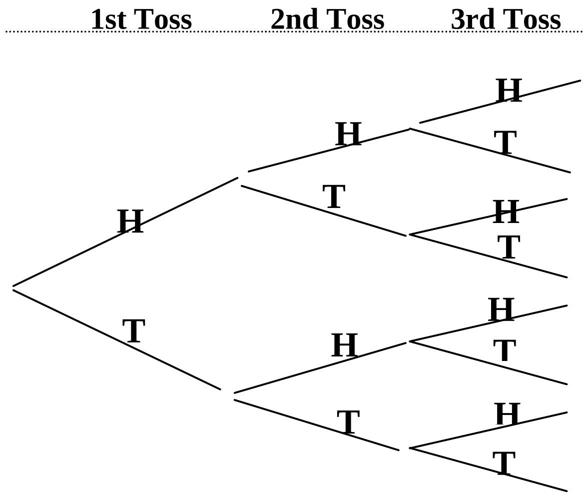
4. The table below shows Fiona's quarterly electricity bills (in £) over a three year period.

Year	**1st quarter**	**2nd quarter**	**3rd quarter**	**4th quarter**
2000	147	105	122	142
2001	118	78	89	115
2002	89	49	55	88

(i) Plot these data on graph paper using suitable axes and scales.
(ii) Calculate appropriate moving averages to smooth the data.
(iii) Plot these averages on the graph obtained in (i) and draw the trend line.
(iv) Use the trend line to estimate Fiona's electricity bill for the first quarter in 2003 **showing clearly where your reading is taken**.
The cost of electricity was 10.8p per unit in 2000, 9.9p in 2001 and 9.4p in 2002.
(v) Are the yearly reductions in Fiona's electricity bill due solely to the reductions in the unit price of electricity? **Show any calculations used to support your answer**. (C.C.E.A. Additional Paper 2 – 2003)

Part 2 - Probability

1. **(a)** Copy and **complete** the probability 'tree' below, which shows a **biased** coin tossed **three** times, giving the **probability** of **3 tails** as $\mathbf{\frac{1}{27}}$.



(b) Using your 'tree' diagram for part **(a)**, find the following probabilities:

(i) A tail from 1 toss.
(ii) A head from 1 toss.
(iii) 3 heads from 3 tosses.
(iv) 2 tails from 2 tosses.
(v) 2 heads from 2 tosses.
(vi) 1 head from 2 tosses.
(vii) 1 tail from 3 tosses.
(viii) 2 heads from 3 tosses.
(ix) No heads from 3 tosses.
(x) No tails from 2 tosses.

2. In their recent exams 75% of a group of 4th year medical students passed paediatrics, 60% passed obstetrics and 50% passed both.

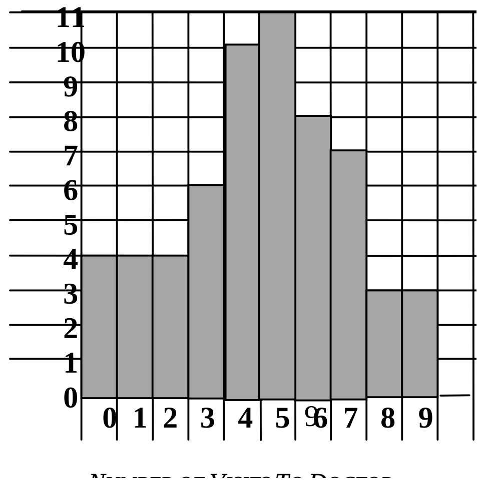
(i) Given that a student passed paediatrics what is the probability that the student also passed obstetrics?

(ii) What percentage of students failed both?

G.C.S.E. (ADDITIONAL) – EXERCISE 3 - ANSWERS

PART 1 - STATISTICS

1.

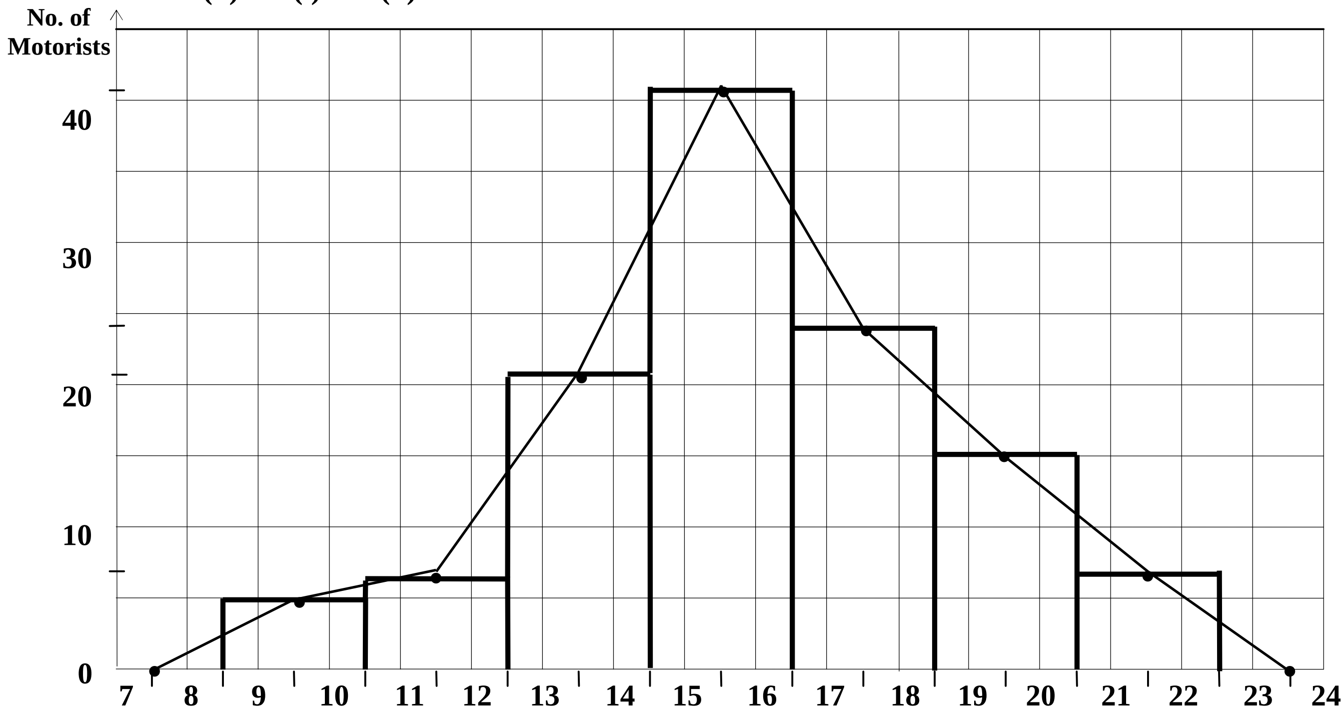


2. **(a)(i)** **Modal class 15-16**.
(ii)

Class Interval	Mid-point	Frequency	Mid-point ´ Frequency
9-10	9.5	5	47.5
11-12	11.5	7	80.5
13-14	13.5	21	283.5
15-16	15.5	41	635.5
17-18	17.5	24	420.0
19-20	19.5	15	292.5
21-22	21.5	7	150.5
	Totals	120	1910.0

Mean $= \frac{1910}{120} = 15.9$ **(to 1 decimal place)**.

(b) **(i)** and **(ii)**.



2.(c) **(i)** **12** **33** **74** **98** **113** **120**

(ii) See **ogive**.

(iii)(1) **15.9** km/l. **(2)** **17.7 - 14.3** = **3.4** km/l.

(iv) **98 – 20** = **78** motorists.

(v)(1) $\frac{12}{120}$ ´ $\frac{100}{1}$ = $8\frac{1}{3}$ %. **(2)** **0.4 ´ 10** = **4** Midgets.

3. We found the **mean** to be **8.55** hours per week.

Score x	**Frequency** f	**Deviation From Mean** (x **- 8.55)**	(x **- 8.55)** ^{2}	**f (**x **- 8.55)** ^{2}
0	**1**	**-8.55**	**73.10**	**73.10**
5	**1**	**-3.55**	**12.60**	**12.60**
6	**1**	**-2.55**	**6.50**	**6.50**
7	**4**	**-1.55**	**2.40**	**9.60**
9	**5**	**+0.45**	**0.20**	**1.00**
10	**6**	**+1.45**	**2.10**	**12.60**
12	**1**	**+3.45**	**11.90**	**11.90**
15	**1**	**+6.45**	**41.60**	**41.60**

Sum of Squared Deviations $\sum f\,(x - 8.55)^2 = 168.90$

Sum of Frequency: $\sum f = 20$

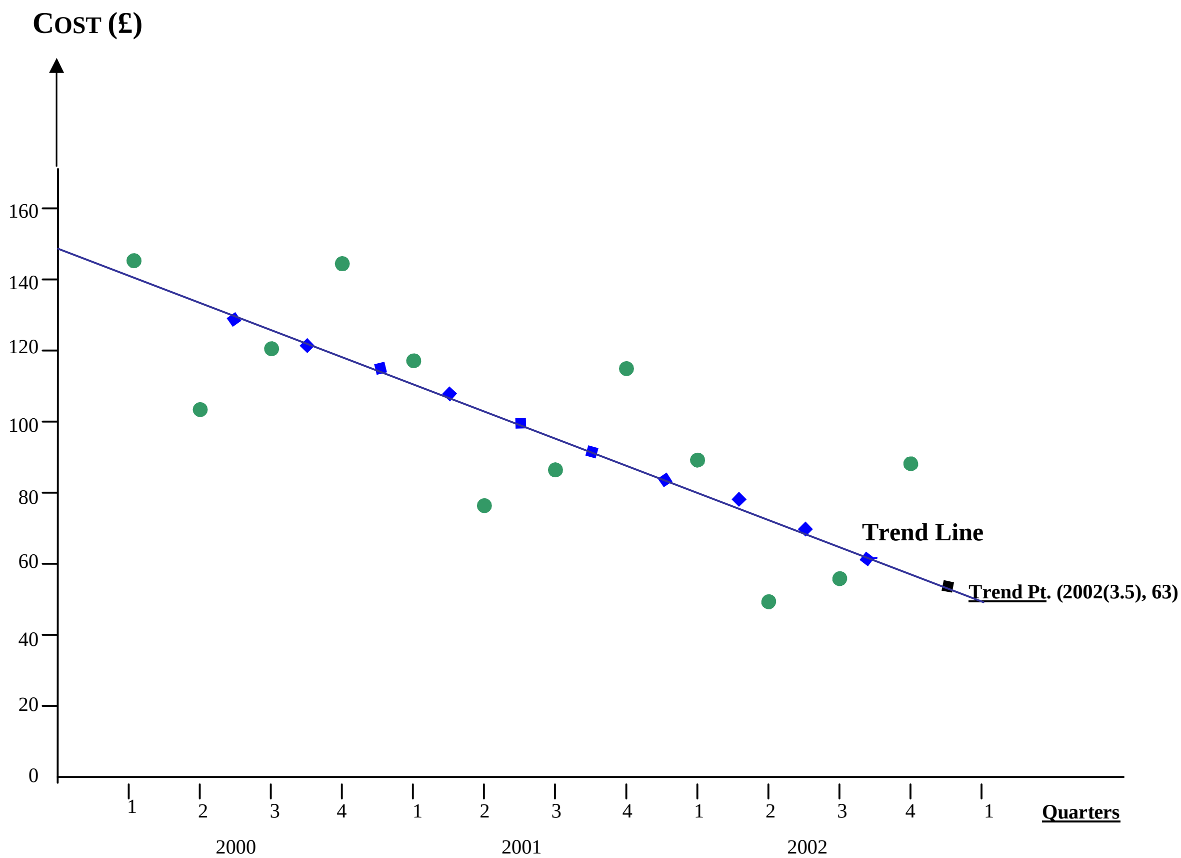
1 Standard Deviation: $= \sqrt{\dfrac{\sum f(x - 8.55)^2}{\sum f}}$

$= \sqrt{\dfrac{168.90}{20}}$

$=$ **2.91** hours

Within the **range** $\mathbf{8.55 \pm 2.91}$ i.e. **5.64** to **11.46** hours we have **16** out of **20**, or **80%** of the scores.

4.(i) (green points) and **(ii)** (blue points)



N.B. The **positioning** of the **moving average** is very important.
The moving average must be plotted at the **mid-point of the data from which it is calculated.**

E.g. The **1st moving average** is computed from quarters **1**, **2**, **3** and **4** of **2000**, giving the **1st moving average mid – way** between quarters **2** and **3** of **2000**, and so on .

Using our trend line, the '**trend point**', the **average** taken from quarters **2**, **3** and **4** of **2002** and quarter **1** of **2003**, gives **£63**. Working backwards, **(49 + 55 + 88 + x)** ÷ **4 = 63** gives **x = £60**, i.e. **£60 electricity bill estimated for 1st quarter of 2003**.

(iii) Using **4 – point** moving averages:

147		
105	129	147 + 105 + 122 + 142 = 516 ÷ 4 = 129
122	121.75	105 + 122 + 142 + 118 = 487 ÷ 4 = 121.75
142	115	122 + 142 + 118 + 78 = 460 ÷ 4 = 115
118	106.75	142 + 118 + 78 + 89 = 427 ÷ 4 = 106.75
78	100	118 + 78 + 89 + 115 = 400 ÷ 4 = 100
89	92.75	78 + 89 + 115 + 89 = 371 ÷ 4 = 92.75
115	85.5	89 + 115 + 89 + 49 = 342 ÷ 4 = 85.5
89	77	115 + 89 + 49 + 55 = 308 ÷ 4 = 77
49	70.25	89 + 49 + 55 + 88 = 281 ÷ 4 = 70.25
55		
88		

(iv) $$\frac{\mathbf{49 + 55 + 88 + x}}{\mathbf{4}} = \mathbf{63}$$

gives $\mathbf{192 + x = 252}$

$\therefore \quad \mathbf{x = 60}$.

This means that our **estimate** of Fiona's electricity bill for the **1st quarter of 2003** is **£60**.

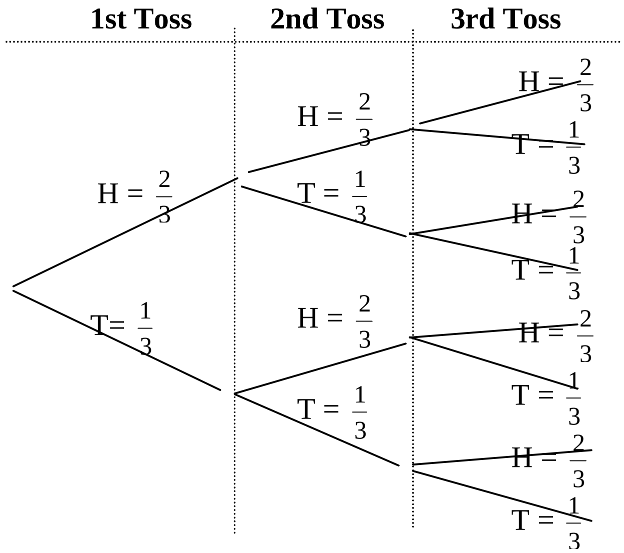
(v)

Year	**Bill**	**Units used**
2000	51600p	**4778**
2001	40000p	**4040**
2002	28100p	**2989**

No. She is using less units of electricity each year.

PART 2 – PROBABILITY

1.



$= \frac{1}{27}$

P (3 Tails) means P (T) **and** P (T) **and** P (T) $= \frac{1}{27}$

$\Rightarrow$ P (T) $\times$ P (T) $\times$ P (T) $= \frac{1}{27}$

$\therefore$ P (T) $= \sqrt[3]{\frac{1}{27}} = \frac{1}{3}$

$\Rightarrow$ P (H) $= 1 - \frac{1}{3} = \frac{2}{3}$.

(b) **(i)** $\frac{1}{3}$ **(ii)** $\frac{2}{3}$ **(iii)** $\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$

(iv) $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$ **(v)** $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$

(vi) P (T) **and** P (H) **or** P (H) **and** P (T)

$\Rightarrow \frac{1}{3} \times \frac{2}{3} + \frac{2}{3} \times \frac{1}{3}$

$= \frac{2}{9} + \frac{2}{9} = \frac{4}{9}$

(vii) P (T) **and** P (H) **and** P (H) $\Rightarrow \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3}$

or P (H) **and** P (T) **and** P (H) $\Rightarrow + \frac{2}{3} \times \frac{1}{3} \times \frac{2}{3}$

or P (H) **and** P (H) **and** P (T) $\Rightarrow$ $+\ \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3}$

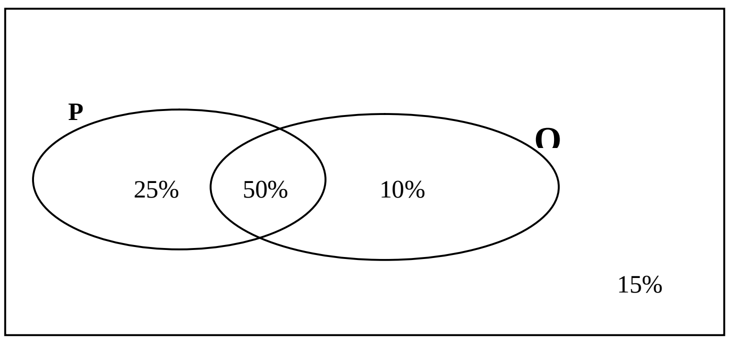
$$= \frac{4}{27} + \frac{4}{27} + \frac{4}{27} = \frac{12}{27} = \frac{4}{9}$$

(viii) Same as **(vii)**, i.e. $\frac{4}{9}$

(ix) P (T) **and** P (T) **and** P (T) $\Rightarrow$ $\frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$

(x) P (H) **and** P (H) **and** P (H) $\Rightarrow$ $\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$

2. **Venn Diagram**:



(i) **P**(**O/P**) $= \dfrac{\mathbf{P(O \cap P)}}{\mathbf{P(P)}} = \dfrac{\mathbf{50\%}}{\mathbf{75\%}} = \dfrac{\mathbf{2}}{\mathbf{3}}$.

(ii) **100% - (25% + 50% + 10%)** **=** **15%**.

SECTION 5

LOGARITHMS AND LOGARITHMIC THEORY

Definition:

A **logarithm** is the **index** (or **power**) to which the **base** must be raised to give a certain **number**.

E.g. $\mathbf{2^3 = 8}$ expresses a relationship amongst the **number 8**, the **base 2** and the **logarithm 3**.

In general, $\mathbf{a^x = b}$ means that **x** is the **logarithm** of **b** to the **base a**.

Since **logarithms** are **indices**, the **properties** of **indices** apply to them:

- **The Multiplication Law**

  - **add the logarithms**

  E.g. $\mathbf{a^x \times a^y = a^{x+y}}$.

- **The Division Law**

  - **subtract** the **denominator logarithm** from the **numerator logarithm**.

  E.g. $\mathbf{a^x \div a^y = a^{x-y}}$.

- **The Powers Law**

  - **multiply** the **logarithms**.

  E.g. $\mathbf{(a^x)^n = a^{nx}}$.

- A **negative logarithm** indicates the **reciprocal**

  i.e. $\mathbf{a^{-x} = \frac{1}{a^x}}$.

  E.g. $\mathbf{2^{-3} = \frac{1}{2^3} = \frac{1}{8}}$,

  which means that **–3** is the **logarithm** of $\mathbf{\frac{1}{8}}$ to the **base 2**

  because $\mathbf{\frac{1}{8} = 2^{-3}}$.

- **Zero Logarithm**

Any **non-zero number** raised to the **logarithm 0** equals **1**.

i.e. $\mathbf{a^0 \quad = \quad 1}$

which means that the **logarithm** of **1** to **base a** is **0**.

Clearly, the **logarithm** of **1** to **any base** is **0**,

Since $\mathbf{10^0 \quad = \quad 1}$,

$\mathbf{20^0 \quad = \quad 1}$,

.
.
.

the **logarithm** of **1** to the **base 10** is **0**,

the **logarithm** of **1** to the **base 20** is **0**

and so on.

- **Fractional Logarithms**

A **fractional logarithm** indicates a **root**.

$$\text{i.e.} \quad \sqrt[n]{a^x} \quad = \quad (a^x)^{\frac{1}{n}} \quad = \quad a^{\frac{x}{n}}.$$

$$\text{E.g.} \qquad \sqrt[5]{32^4} \quad = \quad 32^{\frac{4}{5}} \quad = \quad 16$$

means that $\quad \log_{32} 16 \quad = \quad \frac{4}{5}$.

Since **32** can be written as $\mathbf{2^5}$ and **16** as $\mathbf{2^4}$, we have:

$\log_{2^5} 2^4 \quad = \quad \frac{4}{5}$, which gives:

$$(2^5)^{\frac{4}{5}} \quad = \quad 2^4 \quad \Rightarrow \quad 2^{5 \times \frac{4}{5}} \quad = \quad 2^4 \quad = \quad 16.$$

- **CHANGE OF BASE**

The previous example demonstrates that it is possible to change base:

I changed from **base 32** to **base 2** in the example but I could have changed to **any base**.
Generally:

$$\mathbf{\log_a x = \frac{\log_b x}{\log_b a}}.$$

When we are required to **solve** an **equation** such as $\mathbf{4^x = 7}$,
where the **unknown** is an **index**, it is convenient to use **logarithms**:

$$\mathbf{4^x = 7}$$

Þ $\mathbf{x}$ times $\mathbf{\log 4 = \log 7}$ **(By the multiplication law)**

\ $$\mathbf{x = \frac{\log 7}{\log 4}}.$$

Using **logs** to **any base**, we have:

$\mathbf{x} = \mathbf{1.4}$ (to 1 decimal place).

- Common Logarithms

Although any **positive number** (other than **1**) can serve as a **base** for logarithms,**10** is the most convenient base because the most **common** number system is based on **10**.

Logarithms to the **base 10** are called **common logarithms**.
The **log button** on scientific calculators is **base 10**.

Using **common logarithms**, generally we have:

$$\mathbf{10^{\log arithm} = Number}.$$

Then we have:

$$\mathbf{10^{-3} = \frac{1}{1000} = 0.001},$$

meaning that $\mathbf{\log_{10} 0.001 = -3}$,

$$\mathbf{10^{-2} = \frac{1}{100} = 0.01},$$

meaning that $\mathbf{\log_{10} 0.01 = -2}$,

$$\mathbf{10^{-1} = \frac{1}{10} = 0.1},$$

meaning that $\mathbf{\log_{10} 0.1 = -1}$,

$$\mathbf{10^{0} = 1},$$

meaning that $\mathbf{\log_{10} 1 = 0}$,

$$\mathbf{10^{1} = 10},$$

meaning that $\mathbf{\log_{10} 10 = 1}$,

and so on.

- Natural Logarithms **(**Not on this course**.)**

Although common logarithms serve as the most convenient type for computation, **natural logarithms** are best for theoretical purposes. In the system of **natural logarithms**, the **base** is:

$$\mathbf{e = 2 + \frac{1}{2} + \frac{1}{2 \times 3} + \frac{1}{2 \times 3 \times 4} + \ldots = 2.71828}\ldots$$

Natural logarithms are most useful in theoretical mathematics because many important **formulae** in the **calculus** take their simplest possible forms using **base e logarithms**.

The **ln button** on a scientific calculator is **base e**.

- **Solving equations where the unknown is an index**

  Equations of the type $\mathbf{5^{x} = 11}$ and $\mathbf{3^{(1-2x)} = 7.1}$ are solved by using logs: see worked examples 2 and 4 which follow.

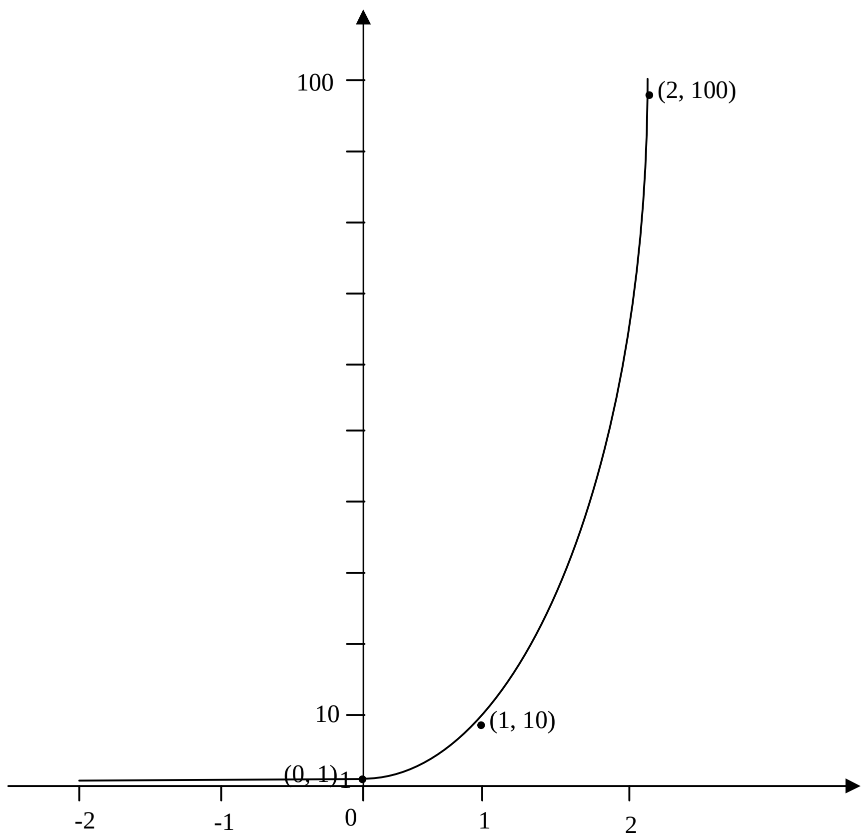
- **Solving equations of the type $y = ax^n$ where a and n are constants**

  Using logs, $\mathbf{y = ax^n}$ can be reduced to $\mathbf{\log y = \log a + n \log x}$, i.e. a **linear** function in the form $\mathbf{y = mx + c}$;
  n is the gradient of the line and **log a** is the **intercept** point on the **log y** – axis: see worked example 6 which follows.

  - **Graphs of the Logarithmic Functions**

The graph of $\mathbf{y = 10^x}$ is an exponential function whose graph is shown below:

Sketch of $\mathbf{y = 10^x}$



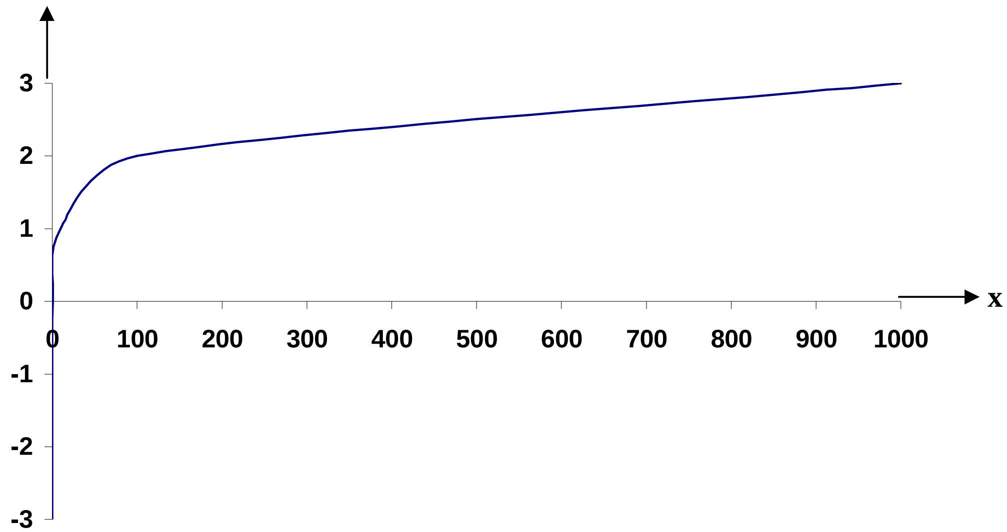
The graph of $\mathbf{y = \log_{10} x}$ has the following table of values for **points (x, y)** on the **curve**.

(Other logarithmic graphs are similar in shape.)

x	**0.001**	**0.01**	**0.1**	**1**	**10**	**100**	**1000**
$\mathbf{y = \log_{10} x}$	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**

The graph of $y = \log_{10}x$ **is shown below.**

Sketch of $y = \log_{10} x$



N.B. Note that $\mathbf{y = \log_{10} x}$ is a reflection of $\mathbf{y = 10^x}$ in the line $\mathbf{y = x}$, i.e. the co-ordinates are reversed:

(-2, 0.01) on $\mathbf{y = 10^x}$ becomes (0.01, -2) on $\mathbf{y = \log_{10} x}$
(-1, 0.1) on $\mathbf{y = 10^x}$ becomes (0.1, -1) on $\mathbf{y = \log_{10} x}$
(0, 1) on $\mathbf{y = 10^x}$ becomes (1, 0) on $\mathbf{y = \log_{10} x}$
(1, 10) on $\mathbf{y = 10^x}$ becomes (10, 1) on $\mathbf{y = \log_{10} x}$
(2, 100) on $\mathbf{y = 10^x}$ becomes (100, 2) on $\mathbf{y = \log_{10} x}$
and so on.

Worked Examples (Logarithms):

1. **Simplify**:

$$\log 8 - 2\log 2 + 3\log 16 - 2\log 32$$

Answer:

$$\log 8 = \log(2^3) = 3\log 2$$
$$\log 16 = \log(2^4) = 4\log 2$$
$$\log 32 = \log(2^5) = 5\log 2$$

$\therefore$ $\log 8 - 2\log 2 + 3\log 16 - 2\log 32$

$= 3\log 2 - 2\log 2 + 12\log 2 - 10\log 2 = 3\log 2$

in its simplest form.

2. **Solve** for **x**:

$$5^{x} = 11$$

Answer:

$$x\log 5 = \log 11$$

\ $x = \dfrac{\log 11}{\log 5}$

Þ $x = 1.5$ (to 1 decimal place).

3. **Solve** for **x** and **y**:

$$10^{x+y} = 1000 \quad \ldots \quad (1)$$
$$\log x - \log y = 1 \quad \ldots \quad (2).$$

Answer:

From **equation** (**1**):

$$10^{x+y} = 1000$$

$\Rightarrow$ $10^{x+y} = 10^{3}$

\ $x + y = 3 \quad \ldots \quad$ **(i)**

From **equation** (**2**):

$$\log x - \log y = 1$$

$\Rightarrow$ $\log_{10}\left(\frac{x}{y}\right) = 1$

$\Rightarrow$ $10^{1} = \dfrac{x}{y}$

\ $x = 10y$ **(By cross-multiplication.)**

or $x - 10y = 0 \quad \ldots \quad$ **(ii)**

Now, we **solve** the **simultaneous equations**:

$x + y = 3 \quad \ldots \quad$ **(i)**

and $x - 10y = 0 \quad \ldots \quad$ **(ii)**.

Eliminating x gives:

(i) – (ii)…$11y = 3$

\ $y = \dfrac{3}{11}$

and $x = 2\dfrac{8}{11}$, which is the **solution**.

4. (a) Solve the equation:

$$\mathbf{3^{(1-2x)} = 7.1}$$

giving your answer correct to 3 decimal places.

(C.C.E.A. Additional Paper 1, 2005)

(b) If $\log_a 81 = 4$ what is the value of **a**?

(c) If log 2 = *p* and log 3 = *q*, express log 36 in terms of *p* and *q*.

Answer:

(a)
$$\mathbf{(1 - 2x)\log 3 = \log 7.1}$$
$$\Rightarrow \mathbf{1 - 2x} = \frac{\log 7.1}{\log 3}$$
$$\Rightarrow \mathbf{1 -} \frac{\log 7.1}{\log 3} = \mathbf{2x}$$
$$\Rightarrow \frac{\mathbf{1 - 1.784155...}}{\mathbf{2}} = \mathbf{x.}$$
$$\backslash \quad \mathbf{x = -0.392} \text{ (to 3 decimal places).}$$

(b)
$$\mathbf{\log_a 81 = 4}$$
$$\Rightarrow \mathbf{a^4} = \mathbf{81}$$
$$\Rightarrow \mathbf{a} = \mathbf{81^{\frac{1}{4}}}.$$
$$\backslash \quad \mathbf{a = 3}.$$

(c)
$$\begin{aligned}\mathbf{\log 36} &= \mathbf{\log (2 \times 2 \times 3 \times 3)}\\ &= \mathbf{\log 2 + \log 2 + \log 3 + \log 3}\\ &= \boldsymbol{p + p + q + q}\\ &= \boldsymbol{2p + 2q} \quad (\text{or } \boldsymbol{2(p+q)}).\end{aligned}$$

5. (a) If $\log_5 a = 2$ what is the value of **a**?

(b) If $\log_5 3 = b$ express $\log_5 45$ in terms of *b*.

(c) Solve the equation:

$$\mathbf{7^{(3x-1)} = 70}$$

giving your answer correct to 3 decimal places.

(C.C.E.A. Additional Paper 1, 2004)

Answer:

(a)

$$\log_5 a = 2$$

$$\Rightarrow \quad 5^2 = a.$$

\ $\quad a = 25.$

(b)

$$\log_5 45 = \log_5 (3 \times 3 \times 5)$$

$$= \log_5 3 + \log_5 3 + \log_5 5$$

$$\log_5 3 = b \quad \text{and} \quad \log_5 5 = 1 \text{ (since } 5^1 = 5)$$

$$\Rightarrow \quad \log_5 45 = b + b + 1.$$

\ $\quad \log_5 45 = 2b + 1.$

(c)

$$7^{(3x - 1)} = 70$$

$$\Rightarrow (3x - 1)\log 7 = \log 70$$

$$\Rightarrow \quad 3x - 1 = \frac{\log 70}{\log 7}$$

$$\Rightarrow \quad 3x - 1 = 2.183294662\ldots$$

$$\Rightarrow \quad x = \frac{2.183294662\ldots}{3}.$$

\ $\quad x = 1.061$ (to 3 decimal places).

6. **Each time a rowing race took place the winning time and the number of crew members in the winning boat were recorded. The results are given in the table below:**

Winning time t (minutes)	**Number of crew** n
13.97	**2**
13.33	**3**
12.90	**4**
11.91	**8**
11.61	**10**

It is believed that a relationship of the form $t = an^b$ exists between t and n, where a and b are constants.

(i) Verify this relationship by drawing a suitable straight line graph. **Label both axes clearly**.

(ii) Hence, or otherwise, obtain values for a and b.

Using the formula $\boldsymbol{t = an^b}$ with the values you obtained for $\boldsymbol{a}$ and $\boldsymbol{b}$ calculate

(iii) the expected winning time for a boat with a crew of 7,

(iv) the least number of crew required to beat a winning time of 12.50 minutes.

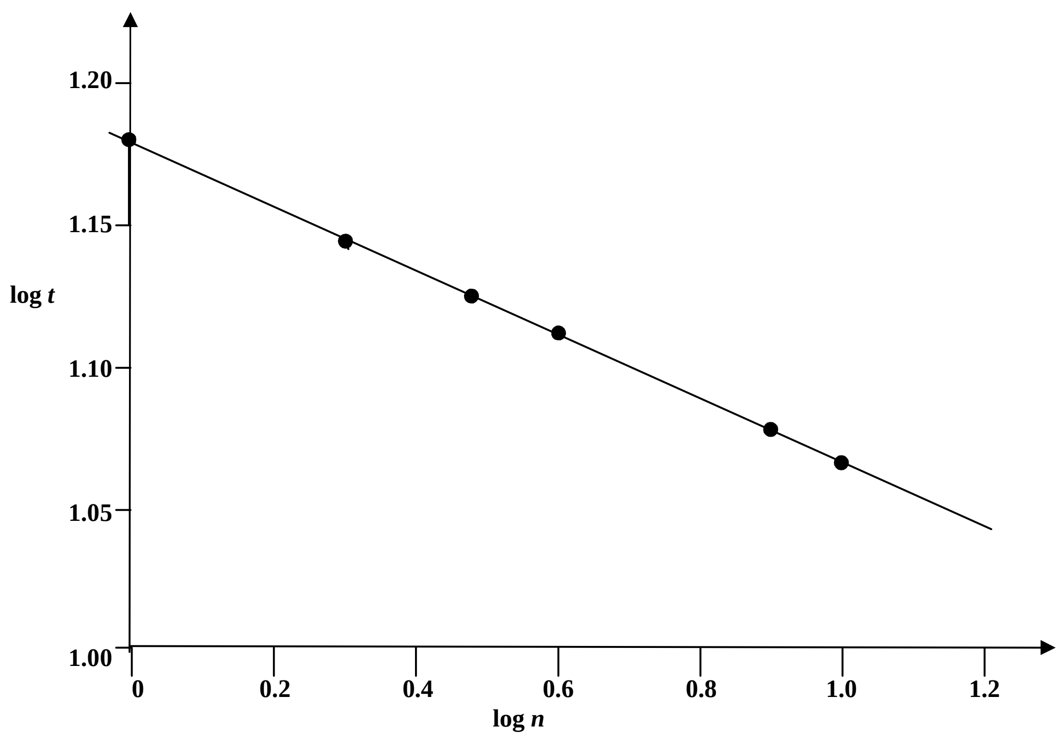
(C.C.E.A. Additional Paper 1, 2004)

Answer:

(i) $\boldsymbol{t = an^b}$

$$\Rightarrow \quad \log t = \log a + b \log n$$

log *n* (x-axis)	**log *t* (y-axis)**
0.301	**1.145**
0.477	**1.125**
0.602	**1.111**
0.903	**1.076**
1	**1.065**



(ii) $\log t = \log a + b \log n$

Using 2 points from the table:

$$1.145 = \log a + 0.301\, b \ \ldots \quad \text{(i)}$$
$$1.065 = \log a + 1.000\, b \ \ldots \quad \text{(ii)}$$

(i) – (ii): $0.08 = -\ 0.699\, b$

$\Rightarrow$ $\boldsymbol{b} = \underline{\mathbf{-\ 0.114}}$.

$\therefore$ $\log a = 1.179$ (substituting in (i))

$\therefore$ $\boldsymbol{a} = \underline{\mathbf{15.116}}$. (i.e. inverse log 1.179)

(iii) $t = an^{b}$

$\Rightarrow$ $t = 15.116(7)^{-0.114}$

$\therefore$ $\boldsymbol{t} = \underline{\textbf{12.11 minutes}}$.

(iv) $t = an^{b}$

$t = 12.50 \Rightarrow 12.50 = 15.116n^{-0.114}$

$$\therefore \quad \frac{12.50}{15.116} = \frac{1}{n^{0.114}}$$

$$\Rightarrow \quad n^{0.114} = \frac{15.116}{12.50}$$

$$\therefore \quad n = \left(\frac{15.116}{12.50}\right)^{\frac{1}{0.114}}$$

$$\Rightarrow \quad n = 5.295\ldots$$

$\therefore$ minimum **<u>6 crew</u>** required for $t < 12.50$ minutes.

GCSE(ADDITIONAL) – EXERCISE 5

1. (a) Solve the equation:

$$\mathbf{0.5^{(1+3x)} = 7}$$

giving your answer correct to 2 decimal places.

(C.C.E.A. Additional Paper 1, 2000)

(b) Solve the equation:

$$\mathbf{\log(5x) - \log\left(\frac{1}{x+1}\right) = 2\log(2x + 2)}.$$

2. (a) Solve the equation:

$$\mathbf{0.3^{-0.6x} = 13}$$

giving your answer correct to 2 decimal places.

(C.C.E.A. Additional Paper 1, 2001)

(b) Solve the equation:

$$\mathbf{\log(x^3) - 2\log x + \log\left(\frac{3}{x^2}\right) = \log(x - 2)}.$$

3. (a) Solve the equation:

$$\mathbf{3^{(2x-1)} = 0.5}$$

giving your answer correct to 3 decimal places.

(C.C.E.A. Additional Paper 1, 2002)

(b) Solve the equation:

$$\mathbf{\log\left(\frac{1}{x-1}\right) + 2\log x = \log(x + 3)}.$$

4. (a) Solve the equation:

$$\mathbf{0.1^{(1-x)} = 7}$$

giving your answer correct to 3 decimal places.

(C.C.E.A. Additional Paper 1, 2003)

(b) Solve the equation:

$$\mathbf{\log x - \log 2 - 2\log(1 - x) = 0}.$$

GCSE(ADDITIONAL) – EXERCISE 5 (ANSWERS)

Answer 1:

(a)

$$0.5^{(1+3x)} = 7$$

$$\Rightarrow \quad (1 + 3x)\log 0.5 = \log 7$$

$$\Rightarrow \quad 1 + 3x = \frac{\log 7}{\log 0.5}.$$

$$\Rightarrow \quad 3x = \frac{\log 7}{\log 0.5} - 1$$

$$\backslash \quad x = -1.27$$ (correct to 2 decimal places).

(b)

$$\log(5x) - \log\left(\frac{1}{x+1}\right) = 2\log(2x + 2).$$

$$\Rightarrow \quad \log \frac{5x}{\frac{1}{x+1}} = \log (2x + 2)^2$$

$$\Rightarrow \quad \log (5x(x + 1)) = \log (2x + 2)^2$$

$$\Rightarrow \quad 5x(x + 1) = (2x + 2)^2$$

$$\Rightarrow \quad 5x^2 + 5x = 4x^2 + 8x + 4$$

$$\Rightarrow \quad 5x^2 + 5x - 4x^2 - 8x - 4 = 0$$

$$\Rightarrow \quad x^2 - 3x - 4 = 0$$

$$\Rightarrow \quad (x - 4)(x + 1) = 0$$

$$\Rightarrow \quad x = 4 \quad \text{or} \quad x = -1$$

$$\backslash \quad x = 4$$ (x = - 1 is impossible as a solution).

Answer 2:

(a)

$$
\begin{aligned}
0.3^{-0.6x} &= 13 \\
\Rightarrow \quad (-0.6x)\log 0.3 &= \log 13 \\
\Rightarrow \quad -0.6x &= \frac{\log 13}{\log 0.3}. \\
\backslash \quad x &= \frac{\log 13}{\log 0.3} \div (-0.6) \\
\backslash \quad x &= 3.55 \text{ (correct to 2 decimal places).}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\log(x^3) - 2\log x + \log\left(\frac{3}{x^2}\right) &= \log(x-2). \\
\Rightarrow \quad \frac{x^3 \text{ ´ } \left(\frac{3}{x^2}\right)}{x^2} &= x - 2 \\
\Rightarrow \quad \frac{3x}{x^2} &= x - 2 \\
\Rightarrow \quad \frac{3}{x} &= \frac{x-2}{1} \\
\Rightarrow \quad x(x-2) &= 3 \\
\Rightarrow \quad x^2 - 2x &= 3 \\
\Rightarrow \quad x^2 - 2x - 3 &= 0 \\
\Rightarrow \quad (x-3)(x+1) &= 0 \\
\backslash \quad x &= 3 \text{ (x = -1 is impossible as a solution).}
\end{aligned}
$$

Answer 3:

(a) $3^{(2x - 1)} = 0.5$

$\Rightarrow \quad (2x - 1)\log 3 = \log 0.5$

$\Rightarrow \quad 2x - 1 = \dfrac{\log 0.5}{\log 3}.$

$\backslash \quad x = (\dfrac{\log 0.5}{\log 3} + 1) \div 2$

$\backslash \quad x = 0.185$ (correct to 3 decimal places).

(b) $\log(\dfrac{1}{x - 1}) + 2\log x = \log(x + 3).$

$\Rightarrow \quad \dfrac{1}{x - 1} \times x^2 = x + 3$

$\Rightarrow \quad \dfrac{x^2}{x - 1} = x + 3$

$\Rightarrow \quad x^2 = (x + 3)(x - 1)$

$\Rightarrow \quad x^2 = x^2 + 2x - 3.$

$\backslash \quad 2x - 3 = 0$

$\Rightarrow \quad x = 1.5.$

Answer 4:

(a) $\quad \mathbf{0.1^{(1-x)} = 7}$

$\Rightarrow \quad \mathbf{(1 - x)\log 0.1 = \log 7}$

$\Rightarrow \quad \mathbf{1 - x = \dfrac{\log 7}{\log 0.1}}$.

$\backslash \quad \mathbf{x = 1 - \dfrac{\log 7}{\log 0.1}}$

$\backslash \quad \mathbf{x = 1.845}$ (correct to 3 decimal places).

(b) $\quad \mathbf{\log x - \log 2 - 2\log(1 - x) = 0}$

$\Rightarrow \quad \mathbf{\log x - \log 2 = 2\log(1 - x)}$

$\Rightarrow \quad \mathbf{\log \dfrac{x}{2} = \log(1 - x)^2}$

$\Rightarrow \quad \mathbf{\dfrac{x}{2} = (1 - x)^2}$

$\Rightarrow \quad \mathbf{x = 2(1 - x)^2}$

$\Rightarrow \quad \mathbf{x = 2(1 - 2x + x^2)}$

$\Rightarrow \quad \mathbf{x = 2 - 4x + 2x^2}$

$\Rightarrow \quad \mathbf{2x^2 - 5x + 2 = 0}$

$\Rightarrow \quad \mathbf{(2x - 1)(x - 2) = 0}$

$\Rightarrow \quad \mathbf{x = 0.5 \quad or \quad x = 2}$

$\backslash \quad \mathbf{x = 0.5}$ (x = 2 is impossible as a solution).

SECTION 6

THE CALCULUS & ITS APPLICATIONS

Part 1 -Differentiation & Integration
Instantaneous Rate of Change
First Principles of Differentiation
Tangent
Normal

The **Calculus** is one of the most useful disciplines in mathematics. It deals with **changing quantities**.

The **Calculus** has **two** main **branches**:

(i) **Differential Calculus**
(ii) **Integral Calculus**.

The central problem of **differential calculus** is to find the **rate** at which a **known**, but **varying quantity**, **changes**.
Integral calculus has the **reverse** problem. It tries to find a quantity when its **rate** of **change** is known.

N.B. There is an **inverse relationship** between **differentiation** and **integration**. This **inverse relationship** is **'The Fundamental Theorem of the Calculus'**: it means that one process undoes the other, as, for example, **addition** of **2** undoes **subtraction** of **2**, or **multiplication** by **3** undoes **division** by **3**, and vice versa.

(I) DIFFERENTIAL CALCULUS

The calculus deals with **functions**. A **function f** is a **correspondence** that associates with **each number x** some **number f(x)**.

E.g. The formula, $\mathbf{y = 2x^2}$ associates with **each number x** some **number y**. If we use **f** to label this function,

then $\mathbf{f(x) = 2x^2}$.

Thus $f(0) = 0$, $f(1) = 2$, $f(2) = 8$, $f(-1) = 2$ and so on.

The **instantaneous rate** of **change** of a function is so important that mathematics has given it a special name, **derivative**,
i.e. the **derived function**.

There are **different notations** used for the **derivative** of a function; it depends on the notation used to represent the **function itself**:

E.g. If $\mathbf{f(x) = 2x^2}$, the **derivative** of **f** is denoted by $\mathbf{f'(x)}$ and for $\mathbf{y = 2x^2}$ (letting $\mathbf{y = f(x)}$), the **derivative** of **y** is denoted by $\mathbf{\frac{dy}{dx}}$.

Remember, $\mathbf{\frac{dy}{dx}}$ is **just notation** to represent the derivative of **y** with respect to **x**:

If $\mathbf{s = 2t^2}$, the derivative using this **Leibnizian notation** is $\mathbf{\frac{ds}{dt}}$, representing the **rate** of **change** of **s** with **t**.
(Baron von Leibniz (1646 – 1716), a German scholar, mathematician and philosopher, shares with Sir Isaac Newton the distinction of developing the theory of the differential and integral calculus, but his notation was adopted, in favour of Newton's.)

The **technique** for **differentiating** a function is as follows:

(1) **Premultiply** by **index**.
(2) **Subtract 1** from index to give **new index**.
(3) The **derivative** of a **constant** is **0**.
(4) **Coefficients** are **not affected** by differentiation.

E.g. 1

$$\mathbf{y = 2x^3 - \frac{2}{3}x^2 + x - 3}$$

$$\Rightarrow \mathbf{\frac{dy}{dx} = 6x^2 - \frac{4}{3}x + 1.}$$

E.g. 2

$$\mathbf{f(x) = x^3 - \frac{1}{3}x^2 - \frac{1}{2}x + 5}$$

$$\Rightarrow \mathbf{f'(x) = 3x^2 - \frac{2}{3}x - \frac{1}{2}.}$$

Differentiating a function is equivalent to finding the **gradient** of the curve. The **gradient** of a curve at **any point** is the same as the **gradient** of the **tangent** to the curve at *that* point. Doing this 'manually' by drawing the **tangent** to the **curve** at the **point** and calculating its **gradient** is neither convenient nor accurate. Since **differentiation** is both convenient and accurate, it is certainly the preferred method for finding the **gradient** of a **curve**. However, being able to perform the operations graphically is fundamental to one's understanding of the processes involved in the calculus, and it also demonstrates what a powerful tool the calculus is; it is so much quicker,

(no need even to draw a graph), and its findings are **100% accurate**.

$\mathbf{\frac{dy}{dx}}$ (or $\mathbf{f'(x)}$, if you prefer), then, gives a **general expression** for the **gradient** of a curve at **any point** on the curve. All that is required to find the **actual gradient** at a **particular point** is to substitute the value of **x** at *that point* into the **general** expression for the **gradient**,

i.e. the derivative, $\mathbf{\frac{dy}{dx}}$.

E.g. 1. If $\mathbf{y = 2x^2 + x - 1}$, find the **gradient** of the curve at the point where $\mathbf{x = 1}$, and, *hence*, find the **equation** of the **tangent** to the curve at the point **(1, 2)**.

$$y = 2x^2 + x - 1$$

Þ $\quad \frac{dy}{dx} = 4x + 1$

$x = 1$ Þ $\quad \frac{dy}{dx} = 4(1) + 1 = 5.$

\ the **gradient** of the curve at the point where $\mathbf{x = 1}$ is **5**.

Q the **gradient** of the **tangent** is also **5**, we have:

$\mathbf{y = mx + c}$ *(Remember the tangent is a straight line.)*

(1, 2) and $\mathbf{m = 5}$ $\Rightarrow$ $\mathbf{2 = 5(1) + c}$

$\Rightarrow$ $\mathbf{2 = 5 + c}$

$\Rightarrow$ $\mathbf{-3 = c}$

\ $\mathbf{y = 5x - 3}$ is the equation of the **tangent** to the curve,

$\mathbf{y = 2x^2 + x - 1}$, at the point **(1, 2)**.

A **normal** is a line **perpendicular** to the **tangent** at the **point** of **tangency**. The **product** of their **gradients** is, therefore, **-1**.

E.g. If a **tangent** has **gradient –2**, the **normal** has **gradient** $\frac{1}{2}$;

if a **tangent** has **gradient** $-\frac{5}{3}$, the **normal** has **gradient** $\frac{3}{5}$.

E.g. 2. Find the **equation** of the **normal** to the curve, $\mathbf{y = 2x^2 + x - 1}$ at the point **(1, 2)**.

Q the **tangent** has **gradient 5**, (already found)

the **normal** has **gradient** $-\frac{1}{5}$.

Then $\mathbf{y = mx + c}$ gives:

$$2 = -\frac{1}{5}(1) + c$$

Þ $\quad c = \frac{11}{5}.$

\ $\quad y = -\frac{1}{5}x + \frac{11}{5}$

or $\mathbf{5y = 11 - x}$ is the **equation** of the **normal** at the point **(1, 2)**.

First principles of Differentiation

Consider two points, **P**(**x**,**y**) and **Q**(**x + h**, **y + k**) on the curve of

$y = ax^2 + bx + c$, where **a**, **b** and **c** are constants.

Then $y + k \quad = \quad a(x + h)^2 + b(x + h) + c$

$\Rightarrow \quad y + k \quad = \quad ax^2 + 2axh + ah^2 + bx + bh + c$ …(i)

and $\quad y \quad = \quad ax^2 + bx + c$ …(ii).

(i) – (ii)… $k \quad = \quad 2axh + ah^2 + bh$.

$\Rightarrow \quad \frac{k}{h} \quad = \quad 2ax + ah + b. \quad (h ¹ 0)$.

$\frac{k}{h}$ is the **gradient** of the **chord** joining the two points,

P(**x**,**y**) and **Q**(**x + h**, **y + k**).

The **limit** of this gradient as $h \to 0$ is $2ax + b$.

This is the **formal proof** that the **gradient** of the **tangent**,

$\frac{dy}{dx}$ to the curve of $y = ax^2 + bx + c$ is $2ax + b$,

as is found by differentiating $ax^2 + bx + c$.

Worked Examples:

Find, **from first principles**, the derived functions of:

1. $y \quad = \quad x^2 - 2x + 1$.
2. $y \quad = \quad x^3 + x^2 - 2x$.
3. $y \quad = \quad \frac{1}{x}$.
4. $y \quad = \quad \frac{1}{x^2}$.

Method:

1. $\mathbf{y} \quad = \quad \mathbf{x^2 - 2x + 1}$.

Consider two points, $(\mathbf{x},\mathbf{y})$ and $(\mathbf{x + h}, \mathbf{y + k})$ on the curve of $\mathbf{y = x^2 - 2x + 1}$.

Then $\mathbf{y + k} \quad = \quad (\mathbf{x + h})^2 - \mathbf{2}(\mathbf{x + h}) + \mathbf{1}$

$\Rightarrow \quad \mathbf{y + k} \quad = \quad \mathbf{x^2 + 2xh + h^2 - 2x - 2h + 1}$ … (i)

and $\quad \mathbf{y} \quad = \quad \mathbf{x^2 - 2x + 1}$ … (ii).

(i) – (ii)…$\mathbf{k} \quad = \quad \mathbf{2xh + h^2 - 2h}$.

$\Rightarrow \quad \frac{\mathbf{k}}{\mathbf{h}}$ (the **gradient** of the **chord** joining the two points) $= \mathbf{2x + h - 2}$. (**h** ¹ **0**).

$\therefore$ the limit of the gradient as **h** tends to **0**, the derived function, $\frac{\mathbf{dy}}{\mathbf{dx}}$ to the curve, $\mathbf{y} \quad = \quad \mathbf{x^2 - 2x + 1}$ is $\quad \mathbf{2x - 2}$.

Method:

2. $\mathbf{y} \quad = \quad \mathbf{x^3 + x^2 - 2x}$.

Consider two points, $(\mathbf{x},\mathbf{y})$ and $(\mathbf{x + h}, \mathbf{y + k})$ on the curve of $\mathbf{y = x^3 + x^2 - 2x}$.

Then $\mathbf{y + k} \quad = \quad (\mathbf{x + h})^3 + (\mathbf{x + h})^2 - \mathbf{2}(\mathbf{x + h})$

$\Rightarrow \quad \mathbf{y + k} \quad = \quad \mathbf{x^3 + 2hx^2 + 3h^2x + hx^2 + h^3 + x^2}$

$\mathbf{+ 2hx + h^2 - 2x - 2h}$ … (i)

and $\quad \mathbf{y} \quad = \quad \mathbf{x^3 + x^2 - 2x}$ … (ii).

(i)–(ii)… $\mathbf{k} \quad = \quad \mathbf{2hx^2 + 3h^2x + hx^2 + h^3 + 2hx + h^2 - 2h}$.

$\Rightarrow \quad \frac{\mathbf{k}}{\mathbf{h}}$ (the **gradient** of the **chord** joining the two points)

$= \quad \mathbf{2x^2 + 3hx + x^2 + h^2 + 2x + h - 2}$. (**h** ¹ **0**).

$\therefore$ the limit of the gradient as **h** tends to **0**, the derived function, $\frac{\mathbf{dy}}{\mathbf{dx}}$ to the curve,

$\mathbf{y} \quad = \quad \mathbf{y = x^3 + x^2 - 2x} \quad$ is $\quad \mathbf{3x^2 + 2x - 2}$.

Method:

3. $\mathbf{y} = \frac{1}{x}$.

Consider two points, (**x**,**y**) and (**x + h**, **y + k**) on the curve of **y** $= \frac{1}{x}$.

Then **y + k** $= \frac{1}{x+h}$ … (i)

and **y** $= \frac{1}{x}$ … (ii).

(i)–(ii)… $\mathbf{k} = \frac{1}{x+h} - \frac{1}{x}$

$\Rightarrow \quad \mathbf{k} = \frac{x - (x+h)}{x(x+h)} = -\frac{h}{x(x+h)}$.

$\Rightarrow \quad \frac{k}{h}$ (the **gradient** of the **chord** joining the two points) $= -\frac{1}{x(x+h)}$ ($\mathbf{h} \neq \mathbf{0}$).

$\therefore$ the **limit** of the **gradient** as **h** tends to **0**,

i.e. the derived function, $\frac{dy}{dx}$, to the curve,

$\mathbf{y} = \frac{1}{x}$ is $-\frac{1}{x^2}$.

Method:

4. $\mathbf{y} = \dfrac{\mathbf{1}}{\mathbf{x^2}}$.

Consider two points, ($\mathbf{x}$,$\mathbf{y}$) and ($\mathbf{x + h}$, $\mathbf{y + k}$)

on the curve of $\mathbf{y} = \dfrac{\mathbf{1}}{\mathbf{x^2}}$.

Then $\mathbf{y + k} = \dfrac{\mathbf{1}}{\mathbf{(x+h)^2}}$ … (i)

and $\mathbf{y} = \dfrac{\mathbf{1}}{\mathbf{x^2}}$ … (ii).

(i)–(ii)… $\mathbf{k} = \dfrac{\mathbf{1}}{\mathbf{(x+h)^2}} - \dfrac{\mathbf{1}}{\mathbf{x^2}}$

$\Rightarrow \quad \mathbf{k} = \dfrac{\mathbf{x^2 - (x+h)^2}}{\mathbf{x^2(x+h)^2}}$

$\Rightarrow \quad \mathbf{k} = \dfrac{\mathbf{-\,2xh - h^2}}{\mathbf{x^2(x+h)^2}}$.

$\Rightarrow \quad \dfrac{\mathbf{k}}{\mathbf{h}}$ (the **gradient** of the **chord** joining the two points) $= \dfrac{\mathbf{-\,2x - h}}{\mathbf{x^2(x+h)^2}}$ ($\mathbf{h} \neq \mathbf{0}$).

$\therefore$ the **limit** of the **gradient** as **h** tends to **0**,

i.e. the derived function, $\dfrac{\mathbf{dy}}{\mathbf{dx}}$, to the curve,

$\mathbf{y} = \dfrac{\mathbf{1}}{\mathbf{x^2}}$ is $-\,\dfrac{\mathbf{2x}}{\mathbf{x^4}}$ i.e. $-\,\dfrac{\mathbf{2}}{\mathbf{x^3}}$.

PART 2

TURNING POINTS, MAXIMUM & MINIMUM VALUES AND POINTS OF INFLEXION
INTEGRAL CALCULUS - AREA UNDER CURVE

Since $\frac{dy}{dx}$ is synonymous with '**gradient'**,

a **positive** $\frac{dy}{dx}$ indicates a **positive** gradient and

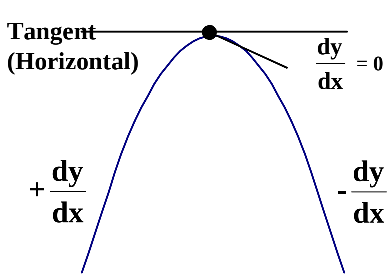
a **negative** $\frac{dy}{dx}$ indicates a **negative** gradient.

When $\frac{dy}{dx}$ changes from positive to negative, or from negative to positive, there is an instant in between where the gradient is **neither negative** nor **positive, i.e.** $\frac{dy}{dx} = 0$**.**

These points on the curve are called
turning points or **stationary points.**

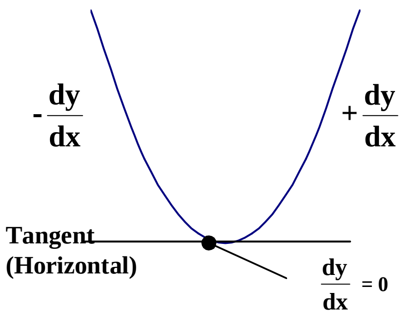
The **gradient** of the curve (or the **tangent** to the curve) at a **turning point** is 0, i.e. the **tangent** is **horizontal**,
and, therefore, parallel to the x-axis.

We have:



MAXIMUM TURNING POINT

Diagram 1



MINIMUM TURNING POINT

Diagram 2

POINTS OF INFLEXION

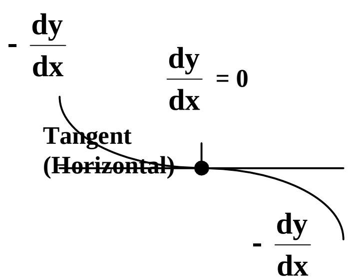


Diagram 3(a)

OR

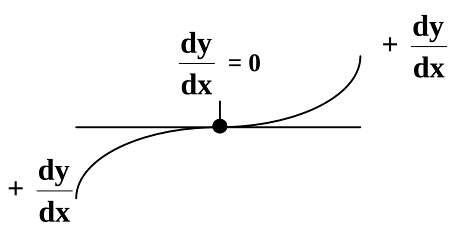


Diagram 3(b)

MAXIMUM & MINIMUM TURNING POINTS

To find the **turning points** on a curve:

(i) Find $\frac{dy}{dx}$.

(ii) Set $\frac{dy}{dx} = 0$.

(iii) Solve for **x**.

If $\mathbf{y = f(x)}$, the **maximum value** of the **function** is the **value** of **y** at the **maximum turning point**. Also, the **minimum value** of the **function** is the **value** of **y** at the **minimum turning point**.

To find the **turning points** on a curve:

(iv) Find $\frac{dy}{dx}$.

(v) Set $\frac{dy}{dx} = \mathbf{0}$.

(vi) Solve for **x**.

E.g. Find the **minimum turning point** on the curve, $\mathbf{y = x^2 - 4}$.

(i) $\frac{dy}{dx} = \mathbf{2x}$.

(ii) $\mathbf{2x = 0}$.

(iii) $\mathbf{2x = 0} \quad Þ \quad \mathbf{x = 0}$.

\ **(0, -4)** is the **minimum turning point** on the curve, $\mathbf{x^2 - 4}$.

To determine whether a turning point is a **maximum**, a **minimum** or a **point** of **inflexion**, you may choose one of **two methods**:

Method (i) Do the $\frac{dy}{dx}$ **test** on the point:

E.g. If we find there is a turning point for $\mathbf{x = 3}$,

i.e. $\frac{dy}{dx} = \mathbf{0}$ when $\mathbf{x = 3}$,

take a value of **x** just before **3**, maybe **2.9**,

and substitute it into $\frac{dy}{dx}$.

Then take another value of **x** just after **3**, maybe **3.1**,

and substitute it into $\frac{dy}{dx}$.

If **x** = **2.9** Þ $\frac{dy}{dx}$ is **positive**, and

if **x** = **3.1** Þ $\frac{dy}{dx}$ is **negative**,

there must be a **maximum turning point** at **x = 3**,

i.e. it looks like **Diagram 1**.

If **x** = **2.9** Þ $\frac{dy}{dx}$ is **negative**, and

if **x** = **3.1** Þ $\frac{dy}{dx}$ is **positive**,

there must be a **minimum turning point** at **x = 3**,

i.e. it looks like **Diagram 2**.

If **x** = **2.9** Þ $\frac{dy}{dx}$ is **negative**, and

if **x** = **3.1** Þ $\frac{dy}{dx}$ is **negative** again,

there must be a **point of inflexion** at **x = 3**,

i.e. it looks like **Diagram 3(a)**.

If **x** = **2.9** Þ $\frac{dy}{dx}$ is **positive**, and

if **x** = **3.1** Þ $\frac{dy}{dx}$ is **positive** again,

there must be a **point of inflexion** at **x = 3**,

i.e. it looks like **Diagram 3(b)**.

Method (ii) Find the **second derivative** ($\frac{d^2y}{dx^2}$ or **f¢¢(x)**).

This means that the function must be **differentiated twice**:

E.g. If $\mathbf{y = x^3}$,

$$\frac{dy}{dx} = 3x^2$$

and $\frac{d^2y}{dx^2} = 6x$.

If we find that there is a **turning point** where **x = 3**, for example, and we wish to know **what kind** of **turning point** it is, we can substitute **x = 3** into the expression for $\frac{d^2y}{dx^2}$, and, then:

$\frac{d^2y}{dx^2} > 0$, i.e. **positive**, implies a **minimum** turning point.

$\frac{d^2y}{dx^2} < 0$, i.e. **negative**, implies a **maximum** turning point.

$\frac{d^2y}{dx^2} = 0$, implies a **point** of **inflexion**

In this case, one would need to do the $\frac{dy}{dx}$ **test**

to get the proper picture:

is it a **Diagram 3(a)**

or a **Diagram 3(b)** type of inflexion?

We shall look at some examples, finding **maximum** and **minimum values**.

E.g. 1 Find the maximum and minimum values of:

$$y = x^3 + \frac{3}{2}x^2 - 18x + 1$$

Þ $\frac{dy}{dx} = 3x^2 + 3x - 18.$

$\frac{dy}{dx} = 0$ at **turning points**:

Þ $3x^2 + 3x - 18 = 0$

(¸ 3) $x^2 + x - 6 = 0$

Þ $(x + 3)(x - 2) = 0$ (by factorisation)

Þ $x = -3$ or $x = 2$.

\ the curve has **turning points** at $x = -3$ and $x = 2$.

When $x = -3$, $y = (-3)^3 + \frac{3}{2}(-3)^2 - 18(-3) + 1 = 41\frac{1}{2}$.

When $x = 2$, $y = 2^3 + \frac{3}{2}(2)^2 - 18(2) + 1 = -21$.

At this stage, we know the curve has **turning points** at $(-3, 41\frac{1}{2})$ and $(2, -21)$.

Next, we need to **test** them to see **what type** of **turning point** each one is:

$\frac{dy}{dx} = 3x^2 + 3x - 18$

Þ $\frac{d^2y}{dx^2} = 6x + 3$ (By **differentiating** $\frac{dy}{dx}$).

$x = -3$ Þ $\frac{d^2y}{dx^2} = 6(-3) + 3 = -18$

Þ **Maximum** turning point at $(-3, 41\frac{1}{2})$.

$x = 2$ Þ $\frac{d^2y}{dx^2} = 6(2) + 3 = +15$

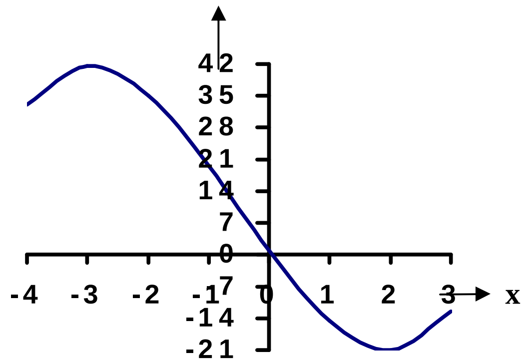
Þ **Minimum** turning point at $(2, -21)$.

N.B. The **maximum** and **minimum values** of **y** are, therefore, $41\frac{1}{2}$ and -21.

Now we can draw a **sketch** of $y = x^3 + \frac{3}{2}x^2 - 18x + 1$.

N.B. **It is helpful to note that the curve crosses the x-axis at the point (0, 1)**.

Sketch of **y** $= x^3 + \frac{3}{2}x^2 - 18x + 1$



E.g. 1 From a rectangular sheet of cardboard of length **30cm** and breadth **20cm**, an equal square of side **xcm** is cut from each corner, so that the remaining flaps can be folded upwards to form a cuboid. Find:

(a) the **volume** of the cuboid in terms of **x** and

(b) the **value** of **x** which would give the **maximum** volume. Also, find the maximum volume.

The diagram looks like this:

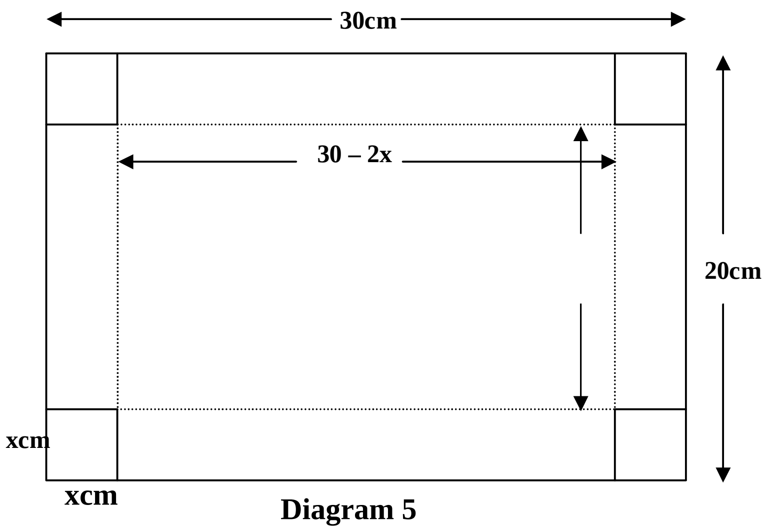


Diagram 5

Volume of **cuboid** = **length $\times$ breadth $\times$ height**

$\Rightarrow$ $V = (30 - 2x) \times (20 - 2x) \times x$

$\Rightarrow$ $V = x(30 - 2x)(20 - 2x)$

$\therefore$ $V = x(600 - 100x + 4x^2)$

$\Rightarrow$ $V = 600x - 100x^2 + 4x^3$ … (a)

$\Rightarrow$ $\frac{dV}{dx} = 600 - 200x + 12x^2$.

$$\frac{dV}{dx} = 0 \quad \text{at turning points}$$

Þ $600 - 200x + 12x^2 = 0$

Þ $150 - 50x + 3x^2 = 0$ $(\div 4)$.

$$3x^2 - 50x + 150 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Þ $x = \dfrac{50 \pm \sqrt{2500 - 1800}}{6}$

\ $x = \dfrac{50 \pm \sqrt{700}}{6}$

\ $x = \dfrac{50 \pm 26.46}{6}$

Þ $x =$ **12.74** or **3.92**.

$x = 12.74$ Þ $V = -315.56\ \text{cm}^3$ … (Reject.)

$x = 3.92$ Þ $V = 1056.31\ \text{cm}^3$.

Since **x = 12.74** gives a **minus** volume, we reject it.

Now, let us **test** the **turning point** at **x = 3.92**:

$$\frac{d^2V}{dx^2} = -200 + 24x.$$

$x = 3.92$ Þ $\dfrac{d^2V}{dx^2} = -200 + 24(3.92) = -105.92$

Þ **Maximum** turning point.

\ **x = 3.92**cm gives the **maximum volume**,

which is **1056.31 cm^3** (as found above.)

Since **integration** is the **reverse process** to **differentiation**, it involves going '**backwards**' from the derivative, $\frac{dy}{dx}$, to the original function, **y**.

E.g. If

$$\frac{dy}{dx} = 3x^2$$

i.e.

$$dy = 3x^2\,dx,$$

we have

$$y = \int 3x^2\,dx.$$

(The symbol, $\int$ is Leibnizian notation for integration).

The **technique** for **integrating** a function is as follows:

(1) **Add 1** to **index**.
(2) **Divide** by **new index**.
(3) Allow for the presence of a **constant** in the original function by adding '**+ c**'; this is known as the '**constant** of **integration'**.
(4) **Coefficients** are **not affected** by integration.

Example

If we look at the function, $y = 2x^3 - \frac{2}{3}x^2 + x - 3$, which was used earlier to demonstrate the differentiation technique, we can see that integrating $\frac{dy}{dx}$ will get us back to **y**.

Given that $\frac{dy}{dx} = 6x^2 - \frac{4}{3}x + 1$ for a certain function **y** and that the point (**0**, **-3**) lies on the curve of **y**, find the **equation** of **y**.

$$\frac{dy}{dx} = 6x^2 - \frac{4}{3}x + 1$$

$$\Rightarrow \quad y = \int (6x^2 - \frac{4}{3}x + 1)dx$$

$$\Rightarrow \quad y = \frac{6x^3}{3} - (\frac{4}{3}x^2 \div 2) + x + c$$

$$\therefore \quad y = 2x^3 - \frac{2}{3}x^2 + x + c.$$

Without knowing a **point** on the curve of **y**, we would *not* be able to find the value of **c**, and we would have to leave it like this. However, we are given that (**0**, **-3**) lies on the curve of **y**, enabling us to find **c**:

$$x = 0,\ y = 3 \quad \Rightarrow \quad -3 = 2(0)^3 - \frac{2}{3}(0)^2 + 0 + c$$

$$\Rightarrow \quad -3 = c$$

$$\therefore \quad y = 2x^3 - \frac{2}{3}x^2 + x - 3$$

is the equation of **y**.

Integrating a function is equivalent to finding the **area under** the **curve**. In GCSE Section 12, we used the Trapezoidal Rule, the Mid-Ordinate rule and Simpson's Rule to **approximate** areas under curves, but integration is **100% accurate**, and, therefore, preferable when the **equation** of the **curve** is **known**.

$\int$ **y dx**, the **indefinite integral**, gives the **general area** between the **curve** of **y** and the **x-axis**, but a function can be integrated and a **definite** area obtained, if this is desired, using **limits** for **x**.

This is known as the **definite integral**. In the definite integral, there is **no need** for **c**, the constant of integration.

Example 1:

Find $\int_1^2 (3x^2 + 1)dx$.

(This means: find the **area** bounded by the curve, **y = 3x² + 1**, the **x-axis** and the lines **x = 1** and **x = 2**.)

The diagram looks like this, with the required area shaded:

$y = 3x^2 + 1$

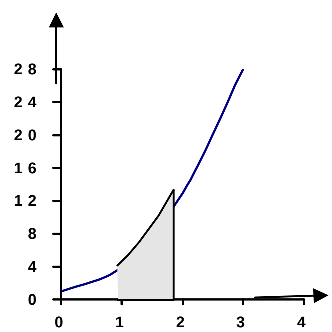


Diagram 6

$$\int_1^2 (3x^2 + 1)dx = \left[x^3 + x\right]_1^2$$

Notice the notation used- [ ] are used once the integration has been done and the limits for x are moved to the right side.

Now, evaluate the integral using the limits, **1** and **2**:

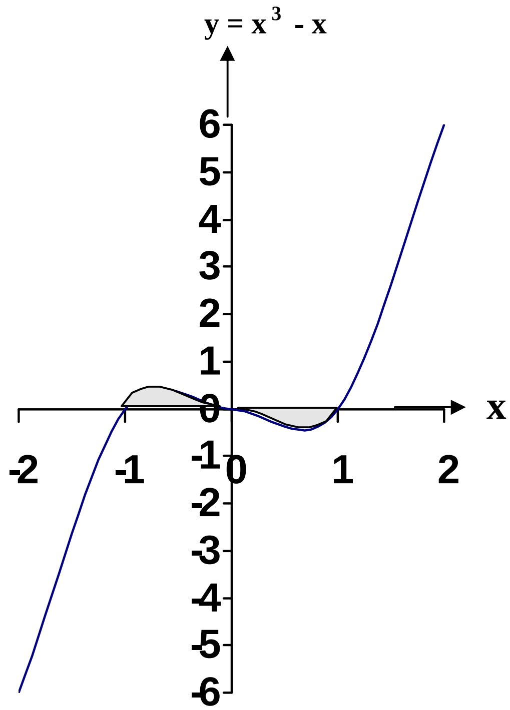
$$\left[2^3 + 2\right] \quad - \quad \left[1^3 + 1\right]$$

$$= \quad 10 - 2$$

$$= \quad 8$$ square units for the shaded region.

Example 2:

Find the **total area** of the **shaded regions** on the following diagram:



Since an **area below** the **x-axis** shows up with a **negative sign** in the integral, we must evaluate the areas of these regions separately and then add them together, **ignoring** the negative sign on the area below the x-axis.

We have, therefore:

$$\int_{-1}^{0}(x^3 - x)dx + \int_{0}^{1}(x^3 - x)dx$$

$$\int_{-1}^{0}(x^3 - x)dx = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0}$$

$$= \left[\frac{0^4}{4} - \frac{0^2}{2}\right] - \left[\frac{(-1)^4}{4} - \frac{(-1)^2}{2}\right]$$

$$= 0 - \left(-\frac{1}{4}\right)$$

$= \qquad \frac{1}{4}$ square unit.

$$\int_0^1 (x^3 - x)dx = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_0^1$$

$$= \left[\frac{1^4}{4} - \frac{1^2}{2}\right] - \left[\frac{0^4}{4} - \frac{0^2}{2}\right]$$

$$= \left[\frac{1}{4} - \frac{1}{2}\right] - [0]$$

$$= -\frac{1}{4} \text{ square unit.}$$

Ignoring the **minus** sign, we have $\frac{1}{4}$ square unit.

$\therefore$ **Total** shaded area $= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ square unit.

Example 3:

Find the **area** between the **line y = x** and the **curve** $\mathbf{y = x^2}$.
The diagram looks like this:

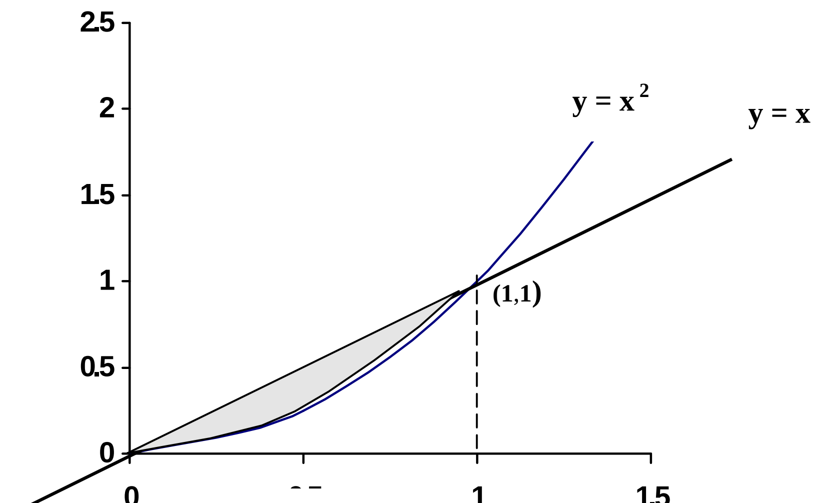


Diagram 8

The **shaded area** on **Diagram 8** is what we are required to find:

(1) The **area** bounded by the **line y = x**, the **x-axis** and the lines **x = 0** and **x = 1** is given by the **triangle**,

base 1 and **perpendicular height 1**,

i.e. area of triangle $= \frac{1}{2}(1)(1) = \frac{1}{2}$ square unit.

(2) The **area** bounded by the **curve** $\mathbf{y = x^2}$, the **x-axis** and the lines **x = 0** and **x = 1** is given by:

$$\int_0^1 x^2 dx$$

$$= \left[\frac{x^3}{3}\right]_0^1$$

$$= \left[\frac{1^3}{3}\right] - \left[\frac{0^3}{3}\right]$$

$$= \frac{1}{3} \text{ square unit.}$$

Since the **shaded area** is the **difference** between (1) and (2), we have:

Shaded area $= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ square unit.

GCSE (ADDITIONAL) EXERCISE 6

1. Differentiate each of the following functions:
   (i) $y = x^3 + x^2 - x + 1$
   (ii) $s = 2t^3 + 3t^2 - 2t - 6$
   (iii) $v = \frac{3}{2}t^2 - \frac{5}{4}t + 1$
   (iv) $y = 2x^4 - 3x^3 + 2x^2 + 6x$
   (v) $y = \frac{1}{2}x^2 - \frac{3}{4}x + \frac{5}{8}$
   (vi) $V = \frac{4}{3}\prod r^3$ (i.e. find $\frac{dv}{dr}$)
   (vii) $f(x) = 2x^3 - 3x^2 + 1$ (i.e. find $f^1(x)$)
   (viii) $f(x) = \frac{1}{2}x^3 - \frac{1}{4}x^2 + \frac{1}{8}$ (i.e. find $f^1(x)$)
   (ix) $f(x) = 6x$ (i.e. find $f^1(x)$)
   x) $f(x) = 10x^3$ (i.e. find $f^1(x)$)

2. Find each of the following **indefinite integrals**:

(i) $\int \sqrt{x}\, dx$ (ii) $\int 3x^2\, dx$ (iii) $\int (x^2 + 1)\, dx$

3. Evaluate each of the following **definite integrals**:

(i) $\int_0^1 x^2\, dx$ (ii) $\int_1^3 \sqrt{(4x)}\, dx$ (iii) $\int_{-2}^1 (2x^2 + 3)\, dx$

4. Find the **equation** of:
   (a) the **tangent**
   (b) the **normal**
   to each of the following curves at the points given:

   (i) $y = 2 + 3x - 2x^2$, at the point where $\boldsymbol{x = 0}$.
   (ii) $y = x^2 + x - 6$, at the point where $\boldsymbol{x = 1}$.
   (iii) $y = 2x^3 + 3x^2 - x + 1$, at the point where $\boldsymbol{x = -1}$.

•

5. Find the **turning points** on the curve $\boldsymbol{y = x^3 + x^2 - x + 1}$. Identify each turning point as a **maximum** or a **minimum**.

6. Find the **total area** between the graph of $\boldsymbol{y = x^3 - 4x}$ and the ***x* - axis**.

7. Find the **area between** the line $\boldsymbol{y = x + 1}$ and the curve $\boldsymbol{y = 2x^2}$.

8. Find the **area bounded** by the curve $\boldsymbol{y = x^2 + 1}$, the ***x* - axis** , the **y-axis** and the line $\boldsymbol{x = 1}$.

PART 2 – THE CALCULUS & ITS APPLICATIONS

1. **(i)** $3x^2 + 2x - 1$ **(ii)** $6t^2 + 6t - 2$ **(iii)** $3t - \frac{5}{4}$

(iv) $8x^3 - 9x^2 + 4x + 6$ **(v)** $x - \frac{3}{4}$ **(vi)** $4\pi r^2$

(vii) $6x^2 - 6x$ **(viii)** $\frac{3}{2}x^2 - \frac{1}{2}x$ **(ix)** 6 **(x)** $30x^2$

2. **(i)** $\frac{2}{3}x^{\frac{3}{2}} + c$ **(ii)** $x^3 + c$ **(iii)** $\frac{x^3}{3} + x + c$

3. **(i)** $\left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$ **(ii)** $\left[\frac{4}{3}x^{\frac{3}{2}}\right]_1^3 = \mathbf{5.6}$ (to 1 d.p.)

(iii) $\left[\frac{2x^3}{3} + 3x\right]_{-2}^1 = \mathbf{15}$

4. **(i)** **(a)** $y = 3x + 2$ **(b)** $x + 3y - 6 = 0$

(ii) **(a)** $y = 3x - 7$ **(b)** $x + 3y + 11 = 0$

(iii) **(a)** $x + y - 2 = 0$ **(b)** $y = x + 4$

5. $\frac{dy}{dx} = 3x^2 + 2x - 1 = 0$ at **turning points**

$\Rightarrow (3x - 1)(x + 1) = 0 \quad \Rightarrow \quad x = \frac{1}{3} \quad$ or $\quad x = -1$

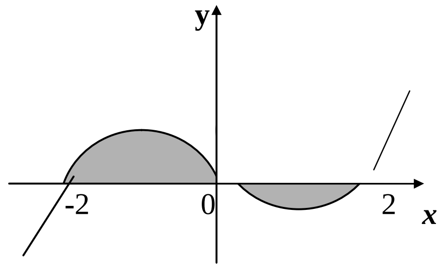
and $y = 1\frac{10}{27}$ or 2 respectively.

$\frac{d^2y}{dx^2} = 6x + 2$

$x = \frac{1}{3} \Rightarrow \frac{d^2y}{dx^2} = +4 \Rightarrow$ **minimum** at $\left(\mathbf{\frac{1}{3}, 1\frac{10}{27}}\right)$;

$x = -1 \Rightarrow \frac{d^2y}{dx^2} = -4 \Rightarrow$ **maximum at (-1, 2)**.

6. $x^3 - 4x = 0 \Rightarrow x(x+2)(x-2) = 0 \Rightarrow x = 0, -2$ or 2.



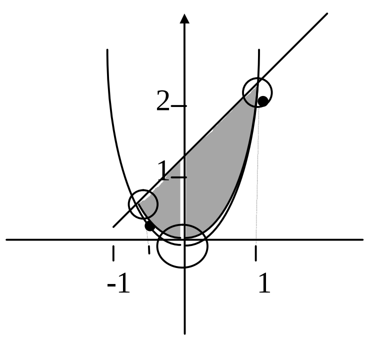
$\int_{-2}^{0}(x^3 - 4x)\,\mathrm{d}x$

$= \left[\frac{x^4}{4} - 2x^2\right]_{-2}^{0} + \left[\frac{x^4}{4} - 2x^2\right]_{0}^{2}$ = **8 sq. units**.

7. $2x^2 = x + 1 \Rightarrow 2x^2 - x - 1 = 0$

$\Rightarrow (2x+1)(x-1) = 0 \Rightarrow x = -\frac{1}{2}$ or 1.

\ $(-\frac{1}{2}, \frac{1}{2})$ and (1, 2) are the **points** of **intersection** between the line and the curve.



Area under **line** : $1\frac{7}{8}$ square units

Area under **curve**: $\int_{-\frac{1}{2}}^{1} 2x^2\,\mathrm{d}x = \frac{3}{4}$ sq. units.

$\therefore$ **Shaded area** = $1\frac{7}{8} - \frac{3}{4} = \frac{9}{8} = \mathbf{1\frac{1}{8}}$ sq. units.

8. $\int_{0}^{1}(x^2 + 1)\,\mathrm{d}x = \left[\frac{x^3}{3} + x\right]_{0}^{1} = \mathbf{1\frac{1}{3}}$ **sq. units**.

VECTORS

- A **vector** is a quantity that has **magnitude** (or modulus) and **direction**.

- Any point **P** (x**, y)** has **position vector** $\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}$, relative to the **origin O (0,0)**.

  The **magnitude** (i.e. modulus or size) of **OP** is $\sqrt{\mathbf{x}^2 + \mathbf{y}^2}$,
  (by **Pythagoras' Theorem**).

- A **displacement vector** or **translation** is equivalent to $\begin{pmatrix} \textbf{Easting} \\ \textbf{Northing} \end{pmatrix}$.

- Vectors may be **added** (or **subtracted**) by **adding** (or **subtracting**) their **components**:

  E.g. $\mathbf{a} = \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} \mathbf{r} \\ \mathbf{s} \end{pmatrix}$

  $$\Rightarrow \quad \mathbf{a} + \mathbf{b} = \begin{pmatrix} \mathbf{p} + \mathbf{r} \\ \mathbf{q} + \mathbf{s} \end{pmatrix}$$

  and $$\mathbf{a} - \mathbf{b} = \begin{pmatrix} \mathbf{p} - \mathbf{r} \\ \mathbf{q} - \mathbf{s} \end{pmatrix}.$$

- A vector may be **multiplied** by a **scalar**:

  E.g. $\mathbf{a} = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} \mathbf{w} \\ \mathbf{z} \end{pmatrix}$

  $$\mathbf{2a} = \begin{pmatrix} \mathbf{2x} \\ \mathbf{2y} \end{pmatrix}$$

  and $$-\frac{3}{2}\mathbf{b} = \begin{pmatrix} -\frac{3}{2}\mathbf{w} \\ -\frac{3}{2}\mathbf{z} \end{pmatrix}.$$

- **Base Vectors – i** and **j**

  i and **j** are *unit vectors* (i.e. vectors of **length 1**)
  in the **positive directions** of the **x** and **y** axes respectively.
  It follows that a point **P** with coordinates **(x , y)** has position vector
  x**i** + y**j** using this notation.

  If **p** represents the position vector **OP**, we have
  $$\mathbf{p} = x\mathbf{i} + y\mathbf{j}.$$

  This is known as '**component**' or '**Cartesian**' form.

DISPLACEMENT, VELOCITY & ACCELERATION

Since the **gradient** of a **displacement/time graph** gives the **velocity**, we **differentiate**,

i.e. $\mathbf{v} = \dfrac{\mathbf{ds}}{\mathbf{dt}}$.

Again, since the **gradient** of a **velocity/time graph** gives the **acceleration**, we **differentiate**,

i.e. $\mathbf{a} = \dfrac{\mathbf{dv}}{\mathbf{dt}}$.

If we wish to go '**backwards**' from **acceleration** to **velocity**, we **integrate**,

i.e. $\mathbf{v} = \int \mathbf{a}\ \mathbf{dt}$.

Going '**backwards'** again from **velocity** to **displacement**, we **integrate**,

i.e. $\mathbf{s} = \int \mathbf{v}\ \mathbf{dt}$.

THE ABOVE INFORMATION **CAN BE SUMMARISED BRIEFLY AS FOLLOWS:**

s **(displacement)**

DIFFERENTIATE **INTEGRATE**

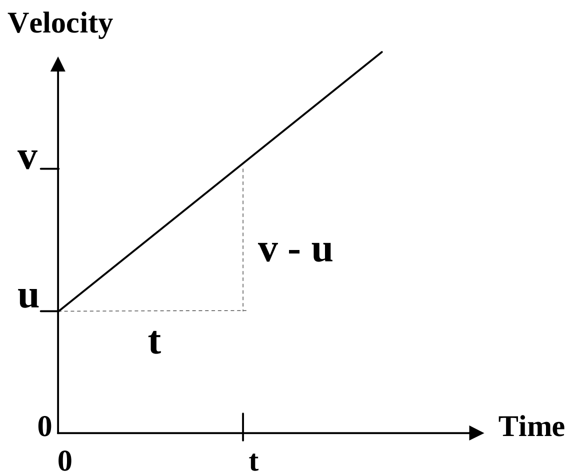
v **(velocity)**

DIFFERENTIATE **INTEGRATE**

a **(acceleration)**

CONSTANT ACCELERATION FORMULAE

(MOTION IN A STRAIGHT LINE)



Let **u** be the **initial velocity** and **v** be the **velocity at time t**.

Since the **gradient** of a **velocity/time graph** gives **acceleration**, we have:

$$\mathbf{a} = \frac{\mathbf{v - u}}{\mathbf{t}}$$

Þ $\mathbf{at} = \mathbf{v - u}$ **(By cross-multiplication)**

or $\mathbf{v} = \mathbf{u + at}$ … **(i)**

Since **area** under a **velocity/time graph** gives **displacement**, we have:

$\mathbf{s} = \frac{1}{2}(\mathbf{u + v})\mathbf{t}$ **(Area of Trapezium)**

Þ $\mathbf{s} = \frac{1}{2}(\mathbf{u + u + at})\mathbf{t}$ **(From (i) above)**

Þ $\mathbf{s} = \frac{1}{2}(\mathbf{2u + at})\mathbf{t}$

or $\mathbf{s} = \mathbf{ut} + \frac{1}{2}\mathbf{at}^2$ … **(ii)**

$\mathbf{v}^2 = (\mathbf{u + at})^2$ **(From (i) above)**

Þ $\mathbf{v}^2 = \mathbf{u}^2 + \mathbf{2uat} + \mathbf{a}^2\mathbf{t}^2$

Þ $\mathbf{v}^2 = \mathbf{u}^2 + \mathbf{2a}(\mathbf{ut} + \frac{1}{2}\mathbf{at}^2)$

Þ $\mathbf{v}^2 = \mathbf{u}^2 + \mathbf{2as}$ … **(iii)(From (ii) above)**

(i), **(ii)** and **(iii)** above are the **CONSTANT ACCELERATION FORMULAE**.

USE OF VECTORS IN KINEMATICS

Vectors in two and three dimensions may be used in kinematics.

Example: (**Constant acceleration**.)

A particle **P** has velocity **v m/s**. When **t = 0**, P has velocity (**3i – 2j**) and is **accelerating constantly** at (**i – 2j**) m/s^2. At time **t = 3**, find:

(i) the speed of P;
(ii) the direction of P.

Worked Answer:

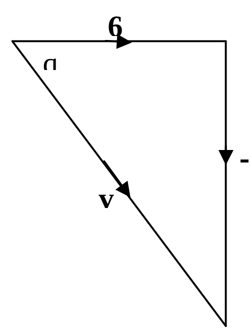
Since the **Constant Acceleration Formulae** can be applied here, we have:

(i) u = 3i – 2j, a = i – 2j, t = 3.
v = u + at
Þ **v = 3i – 2j + 3(i – 2j) = 6i – 8j**.

Since **speed** is the **magnitude** of the **velocity**, we have:

Speed = $\sqrt{6^2 + 8^2}$ = **10m/s**.

(ii) The diagram looks like this:



Tan q = $\frac{8}{6}$ Þ q = **53.1**°.

When **t = 3**, P is moving at **53.1**° **clockwise** to the direction of **i**.

Displacement, Velocity & Acceleration using the Calculus

Example 1:

A particle starts from rest and moves in a straight line, accelerating uniformly at 5m/s^2.
Find its velocity after 6 seconds.

Since the acceleration is constant, we may use the constant acceleration formula:

v = u + at.
We have: **u = 0** ; **a = 5** ; **t = 6**
∴ **v = 0 + 5(6)**
∴ **v = 30**.

This means that the velocity after 6 seconds is **30**m/s.

Example 2:

A ball is thrown vertically upwards from a point P with velocity 15m/s.
Taking g = 10m/s^2, find:

(i) the maximum height reached by the ball;
(ii) the velocity after **3** seconds;
(iii) the height of the ball above P after **3** seconds.

(i) (**NB**: MOTION CAN BE IN EITHER DIRECTION ALONG A STRAIGHT LINE. ONE DIECTION IS TAKEN TO BE POSITIVE AND THE OTHER NEGATIVE)

Since the acceleration due to gravity is 10m/s^2 **downwards**, we use –10m/s^2 for the acceleration **upwards**.
(Sometimes g is taken to be 9.8m/s^2 or 9.81m/s^2).

Note that, since this acceleration is **constant** we may use the **constant acceleration formulae**.

$$\mathbf{v = u + at.}$$

We have: $\mathbf{u = 15}$; $\mathbf{a = -10}$

At the maximum height, $\mathbf{v = 0}$.

$\therefore \quad \mathbf{0 = 15 - 10t}$

$\Rightarrow \quad \mathbf{t = 1.5}$

Using $\mathbf{s = ut + \frac{1}{2}at^2}$

$\Rightarrow \quad \mathbf{s = 15(1.5) + \frac{1}{2}(-10)(1.5^2)}$

$\Rightarrow \quad \mathbf{s = 11.25m}$.

This means that the maximum height reached is **11.25**m.

(ii) Using $\mathbf{v = u + at}$ with $\mathbf{t = 3}$, we have:

$\mathbf{u = 15}$; $\mathbf{a = -10}$; $\mathbf{t = 3}$
giving $\mathbf{v = 15 - 10(3)}$

$\therefore \quad \mathbf{v = -15}$

This means that the ball is **falling** at **15**m/s after **3** seconds.

(iii) Using $\mathbf{s = ut + \frac{1}{2}at^2}$,
we have:

$$\mathbf{s = 15(3) + \frac{1}{2}(-10)(3^2)}$$

giving $\mathbf{s = 45 - 5(9)}$
$\therefore \quad \mathbf{s = 0}$
This means that the ball is **0**m above the point P after **3** seconds, i.e back at P.

Example 3:

If the **displacement**, **s**, of a moving body after time **t** seconds is given by the equation:

$$\mathbf{s} = \frac{2}{3}\mathbf{t}^3 - \mathbf{t}^2 - 4\mathbf{t} + 6 \text{, find:}$$

(i) the velocity, **v**, after **3** seconds.
(ii) the acceleration, **a**, after **3** seconds.
(iii) when the velocity is zero.
(iv) when the acceleration is zero.

$$\mathbf{s} = \frac{2}{3}\mathbf{t}^3 - \mathbf{t}^2 - 4\mathbf{t} + 6$$

(i) $\mathbf{v} = \frac{ds}{dt} = 2t^2 - 2t - 4$

$t = 3 \Rightarrow v = 2(3)^2 - 2(3) - 4$

$\Rightarrow v = 8\text{m/s}$ after **3**seconds.

(ii) $\mathbf{a} = \frac{dv}{dt} = 4t - 2$

$t = 3 \Rightarrow a = 4(3) - 2$

$\Rightarrow a = 10\text{m/s}^2$ after **3** seconds.

(iii) **Q** ‘**when**’ means ‘at what time’,
we need to find **t** when **v** = **0**:

$2t^2 - 2t - 4 = 0$

$(\div 2) \Rightarrow t^2 - t - 2 = 0$

$\Rightarrow (t - 2)(t + 1) = 0$ (By factorisation)

$\Rightarrow t = 2$

$\Rightarrow$ the **velocity** is **zero** after **2** seconds.

(This means, of course, that the body is **at rest** after **2** seconds.)

(iv) We need to find **t** when **a** = **0**:

$4t - 2 = 0$

$\Rightarrow t = \frac{1}{2}$

$\Rightarrow$ the **acceleration** is **zero** after $\frac{1}{2}$ second.

Example 4:

The velocity, **v**, of a moving body is $(\mathbf{2t^2 + 1})$m/s, after a time, **t** seconds.
Find:
(i) the **displacement** after **2** seconds.
(ii) the **initial velocity**.
(iii) the **acceleration** after **1** second.

(i) $s = \int v\,dt + c$

$\Rightarrow \quad s = \int (2t^2 + 1)\,dt + c$

$\therefore \quad s = \dfrac{2t^3}{3} + t + c$

$t = 0,\ s = 0 \quad \Rightarrow \quad 0 = \dfrac{2(0)^3}{3} + 0 + c$

$\Rightarrow \quad c = 0$

$\therefore \quad s = \dfrac{2t^3}{3} + t$.

$t = 2 \quad \Rightarrow \quad s = \dfrac{2(2)^3}{3} + 2$

$\Rightarrow \quad s = 7\frac{1}{3}$ m.

(ii) The **initial velocity** is at **t = 0**.

$\therefore \quad v = 2(0)^2 + 1 = 1$ **m/s**.

(iii) $a = \dfrac{dv}{dt} = 4t$

$t = 1 \quad \Rightarrow \quad a = 4\text{m/s}^2$ after **1** second.

Example 5:

The acceleration**, a**, of a moving body at the end of **t** seconds from the start of motion is **(7 – t)m/s**^{2}.
Find:

(i) the velocity
(ii) the displacement at the end of **3** seconds, if the initial velocity is **5m/s**.

(i) $v = \int a\ dt + c$

$\Rightarrow \quad v = \int (7 - t)\,dt + c$

$\Rightarrow \quad v = 7t - \dfrac{t^2}{2} + c.$

$t = 0,\ v = 5 \quad \Rightarrow \quad 5 = 7(0) - (\dfrac{0^2}{2}) + c$

$\Rightarrow \quad c = 5$

$\therefore \quad v = 7t - \dfrac{t^2}{2} + 5.$

$t = 3 \quad \Rightarrow \quad v = 7(3) - (\dfrac{3^2}{2}) + 5$

$\therefore \quad v = 21\frac{1}{2}$ **m/s** after **3** seconds.

(ii) $s = \int v\,dt + c$

$$\Rightarrow \quad s = \int (7t - \frac{t^2}{2} + 5)dt + c$$

$$\Rightarrow \quad s = \frac{7t^2}{2} - \frac{t^3}{6} + 5t + c$$

$$t = 0, s = 0 \Rightarrow \quad 0 = \frac{7(0)^2}{2} - \frac{(0)^3}{6} + 5(0) + c$$

$$\Rightarrow \quad c = 0$$

$$\therefore \quad s = \frac{7t^2}{2} - \frac{t^3}{6} + 5t.$$

$$t = 3 \quad \Rightarrow \quad s = \frac{7(3)^2}{2} - \frac{(3)^3}{6} + 5(3)$$

$$\Rightarrow \quad s = \frac{63}{2} - \frac{27}{6} + 15$$

$$\therefore \quad s = \mathbf{42m} \quad \text{after } \mathbf{3} \text{ seconds.}$$

Example 6:

The velocity, **v**, of a moving body, starting from rest, is $\mathbf{2t^2 + t}$ m/s, after a time **t** seconds.

Find:
(i) the **displacement** after **1** second,
(ii) the **initial** velocity and
(iii) the **acceleration** after **2** seconds.

(i)

$$s = \int (2t^2 + t)dt = \frac{2t^3}{3} + \frac{t^2}{2} + c$$

$$t = 0, s = 0 \Rightarrow c = 0$$

$$\therefore \; s = \frac{2t^3}{3} + \frac{t^2}{2}$$

$$t = 1 \Rightarrow s = \frac{2}{3} + \frac{1}{2} = \mathbf{1\frac{1}{6}\,m}.$$

(ii) $t = 0 \Rightarrow v = 0$

(iii)

$$a = \frac{dv}{dt} = 4t + 1$$

$$t = 2 \Rightarrow a = \mathbf{9\ m/s^2}.$$

FORCE

A **force** is a vector quantity that causes a **change** in the **state of motion** of a body.
A body in motion can change its velocity or direction ***only if*** a resultant force acts upon it.
The **unit** of force is the **newton** (**N**).
A force of 1**N** produces an acceleration of 1**m/s** ^{2} in a body of mass 1**kg**.
The **weight** of a body is the force exerted upon it by **gravity** (g = 9.81 m/s ^{2}).

Generally, the **weight** of a body of mass **m kg** is **mg N**.

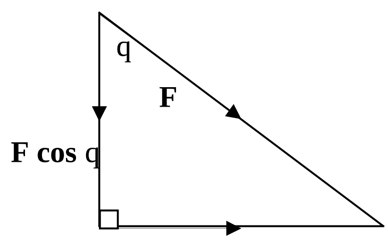
E.g. A person with a ***mass*** of **60 kg** has a ***weight*** of approximately **600 N**
(g is often taken as 10 m/s ^{2}).

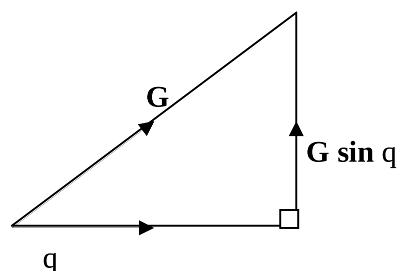
Resolving Forces

Forces can be **resolved** into **two components** at **right-angles** to each other when a **right-angled triangle** is constructed around the force, making the **force** the **hypotenuse**. The forces are represented in **magnitude** and **direction** by the sides of the triangle in each case, using **addition of vectors**.

Look at how the forces **F** and **G** below are **each** resolved into **two components**:

(N.B. It is important to note the **direction of the arrow** in the force being resolved, as each diagram represents **addition of vectors**.)



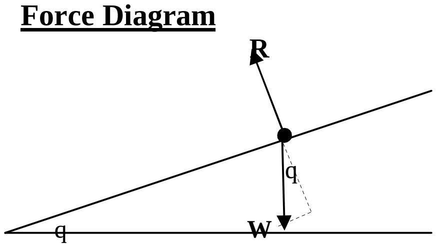


Inclined Plane

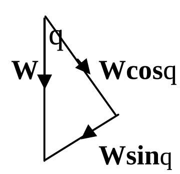
In a force diagram, a **force** can be ***replaced*** by its **components**.
(Do not show the force and its components on the same diagram.)

This is particularly helpful when dealing with an object on an inclined plane.

Force Diagram



Vector Diagram



Look at the **force diagram** and the **vector diagram** above.

The **force diagram** shows a body of weight **W** resting on a smooth plane, which is inclined at an angle q to the horizontal.

R, the **force** that the **plane exerts** on the body, acting at **right-angles** to the plane, is called the **normal reaction**.

The weight **W** can be resolved into components **perpendicular** and **parallel** to the plane, as demonstrated on the **vector diagram**. This diagram shows that the component of **W** acting *parallel* to the plane is **W sin** q and the component of **W** *perpendicular* to the plane is **W cos** q.

(These are derived from basic trigonometry on the right-angled triangle:

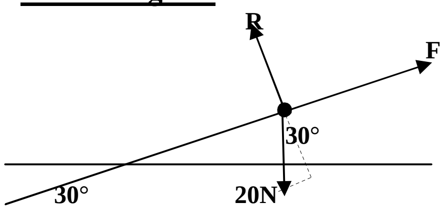
Sin q = $\frac{\textbf{parallel}}{\textbf{W}}$ Þ **parallel** = **W sin** q

Cos q = $\frac{\textbf{perpendicular}}{\textbf{W}}$ Þ **perpendicular** = **W cos** q.

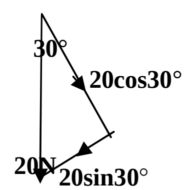
Example:

A body of weight **20N** is **at rest** on a plane, which is inclined at an angle of **30°** to the horizontal. The body is held in place by 2 forces, **F** and **R**, as shown in the **force diagram** below. Find the 2 forces, **F** and **R**.

Force Diagram



Vector Diagram



The force **F** is given by the **parallel** component:
⇒ F = 20 sin 30° = **10N** in the **direction of F**.

The force **R** is given by the **perpendicular** component:
⇒ R = 20 cos 30° = **17.3N in the direction of R**.

(NOTE: THE FORCES MUST BALANCE IN EACH DIRECTION)

Resultant Force

Two or more forces acting at a point have the same effect as a **single force**, found by **vector addition**.
This single force is called the **resultant** of the forces.

Example:

Find the magnitude and direction of the resultant force, **F** N of the forces **P** = (–2i + 3j) N, **N** = (i + 2j) N and **M** = (4i – j) N.

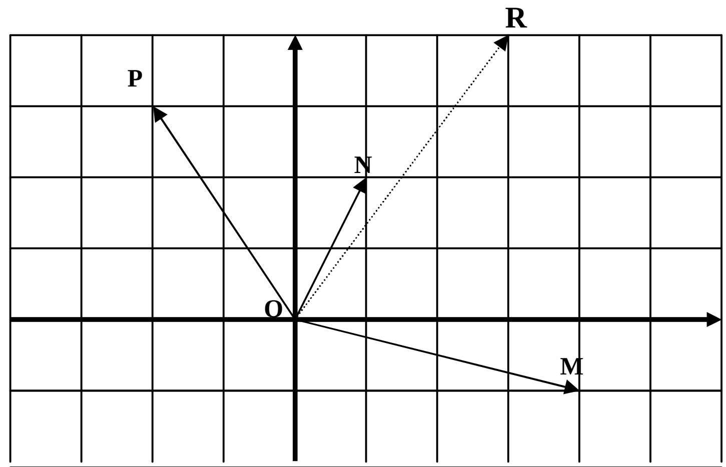
The **resultant force F** is given by:

$$(-2i + 3j) + (i + 2j) + (4i - j) = \mathbf{3i + 4j}.$$

The **magnitude**, $|\mathbf{F}| = \sqrt{\mathbf{3^2 + 4^2}} = \mathbf{5}$.

The **direction** is $\mathbf{tan}^{-1}\left|\frac{\mathbf{4}}{\mathbf{3}}\right|$ = **53.1°** from the **positive direction** of the **x-axis**.

(The resultant **R** is the dashed line........... on the diagram below.)



Equilibrium

If a body is **not moving**, then the **resultant force** in any direction must be **zero**.

Hence, if **R** is the **resultant** of three forces **P**, **N** and **M**, then **–R** added to **P**, **N** and **M** will produce equilibrium.

Look again at the diagram above.

We found that **P + N + M** = **3i + 4j** = **R**

Þ **-3i – 4j** = **-R**.

We have:

(–2i + 3j) + (i + 2j) + (4i – j) + (-3i – 4j) = **0i** + **0**j

⇒ **no movement** from **O**

⇒ forces are in **equilibrium**.

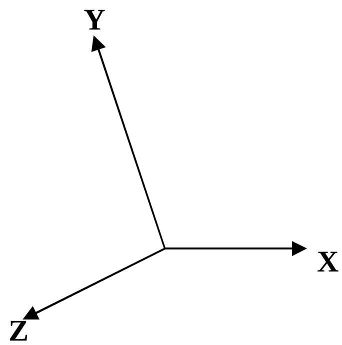
Triangle of Forces

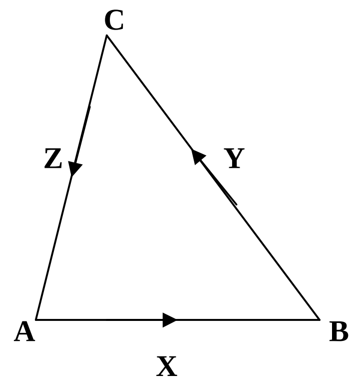
If three forces acting at a point can be represented by the sides of a triangle and the **arrows** on the sides of the triangle indicating the **directions** of the forces are all in the **same sense**, then these forces are in **equilibrium**.

Conversely, if three forces acting at a point are in equilibrium, they can be represented by the sides of a triangle.

The triangle **ABC**, shown on the next page, is said to be a **Triangle of Forces** for the three forces, **X**, **Y** and **Z**.

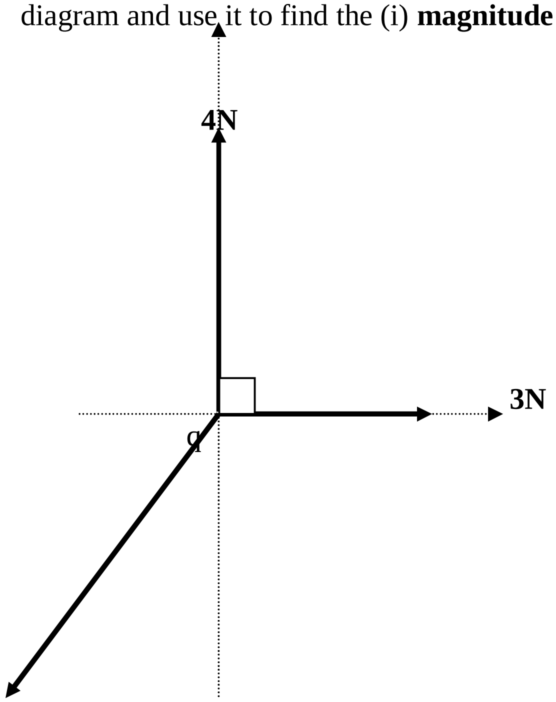
Triangle of Forces – X, Y and Z

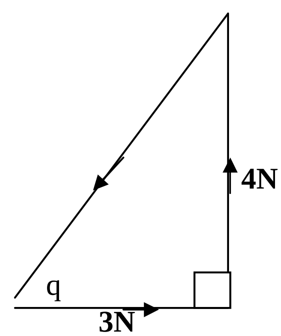




Example:

Given that the three forces shown in the diagram below are in **equilibrium**, draw a **scale** diagram and use it to find the (i) **magnitude** and (ii) **direction** of **F**.





Method:

Draw a horizontal line 3 units long, followed by a perpendicular line, 4 units long and complete the triangle. Measure the third side.

(i) The magnitude is **5**N.

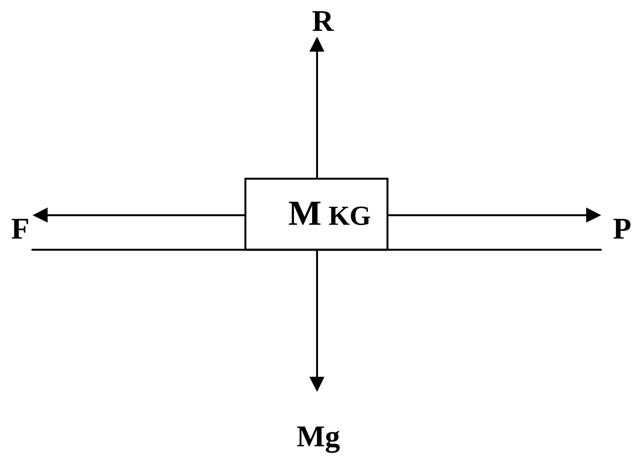
(ii) The direction, q, is $\text{Tan}^{-1}\left(\frac{\mathbf{4}}{\mathbf{3}}\right) \quad = \quad \mathbf{53.1°}$.

(**N.B. The sides of the triangle must be drawn parallel to the vectors of the forces**.)

FRICTION

If a body of mass **M** kg rests on a horizontal surface, and a horizontal force of **P** N is applied to the body, **equal** and **opposite** forces act on the body and on the plane at **right-angles** to the surfaces in contact. *(Newton's Third Law).*

$$\therefore \quad \mathbf{R} = \mathbf{Mg}$$



The force, **F**, *opposing* the motion of the body, is called the **frictional force**.
The frictional force acts in the **opposite** direction to the motion and is **parallel** to the surfaces that are in contact.

With perfectly **smooth** surfaces, there is **no** frictional force, i.e. **F = 0**,
so the body would move no matter how small is the applied force **P**.
With **rough** surfaces, the body will move *only if* **P**, the applied force,
is **greater** than **F**, the frictional force.

The magnitude of the frictional force depends on the roughness of the surfaces

and the force **P**, which is trying to move the body.
For **small** values of **P**, there is no movement of the body and **F = P**.
As the applied force **P** increases, the frictional force **F** increases until **F** reaches a **maximum value F(max)**, beyond which it cannot increase.
At this point, the body is in a state of **limiting equilibrium**, and
is *on the point* of moving.
If **P** is increased again to a value $\mathbf{P}^1$, the body will move, since **F** has already reached its maximum value and the equilibrium is broken.

Note that **F(max)** will only act if:
- The body is in a state of limiting equilibrium
  or
- It is already moving.

N.B. F is only as large as is necessary to prevent motion.)

Now applying the equation of motion, **F = ma**, we have:

$$\mathbf{P}^1 - \mathbf{F(max)} = \text{mass} \times \text{acceleration}$$

Coefficient of Friction

The magnitude of **F(max)** is a **fraction** of the normal reaction **R**.
This **fraction** is called the **coefficient** of **friction** and is denoted by m for the two surfaces in contact.
We have: F(max)= μR.

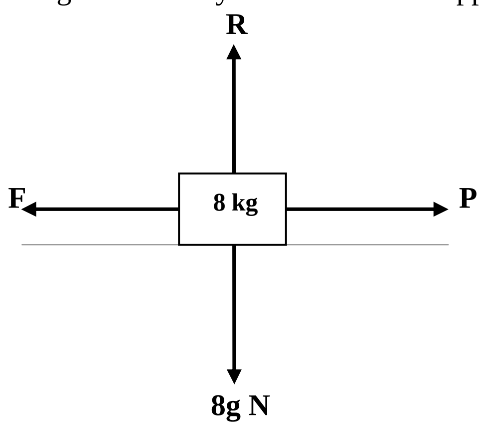
Laws of friction

The frictional force:

1. acts parallel to the surfaces that are in contact and in a direction opposing the motion of one body across the other;
2. is only as large as is necessary to prevent this motion;
3. has a maximum value μR, where R is the normal reaction between the surfaces that are in contact;
4. is taken as having its maximum value μR when motion takes place;
5. depends on the type of the surfaces in contact, not on the area of contact.

Example:
An object of mass **8 kg** rests on a rough horizontal surface.
The coefficient of friction between the object and the surface is **0.4**.
Calculate: (a) the frictional force acting on the object when a horizontal force **P =** (i) **15 N** (ii) **31.36 N** (iii) **80 N** is applied to the object.
(b) the magnitude of any acceleration that applies.

Method:



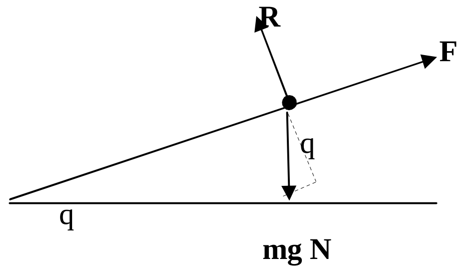
There is no motion vertically, so: Resolving vertically gives R = 8g N = 78.4 N.
Then: μR = 0.4×78.4 = 31.36 N
(i) Resolving horizontally: F = P $\Rightarrow$ **F = 15 N**
(Note that there is **no motion** since **P** < m**R**.)
(ii) F = P $\Rightarrow$ **F = 31.36 N**
(This time the object is **on the point** of **moving** since **P =** m**R**).
(iii) F = P $\Rightarrow$ **F = 80 N** (This time the object is **in motion** since **P** > m**R**).
Applying **F = ma**: F - μR = mass $\times$ acceleration
$\Rightarrow$ 80 – 31.36 = $8 \times$ acceleration
$\Rightarrow$ 48.64 ÷ 8 = 6.08
$\Rightarrow$ **acceleration** = **6.08m/s^2**.

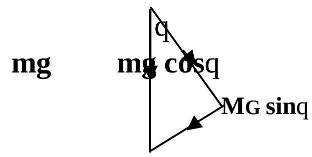
Rough Inclined Plane

(i) <u>**m = tan q in position of limiting equilibrium**</u>

(ii) <u>**m < tan q for motion to take place**</u>

Force Diagram **Vector Diagram**





If a body of mass m kg is **on the point of slipping** down a rough plane which is inclined at an angle q to the horizontal, it can be shown that <u>m = **tan** q</u>,
and when the body is **in motion**, <u>m < **tan** q</u>,
where m is the **coefficient of friction**.

Method: See the force and vector diagrams above.

Resolving parallel to the plane: $F = mg \sin \theta$.

Resolving perpendicular to the plane: $R = mg \cos \theta$.

In the position of **limiting equilibrium**: **F =** m**R**

$\Rightarrow \quad mg \sin \theta = \mu\, mg \cos \theta$

$\Rightarrow \quad \dfrac{\mathbf{mg\,sin}\ J}{\mathbf{mg\,cos}\ J} = m$

$\therefore \quad$ **tan** q = m.

For motion to take place down the plane, **F**, the force acting down the plane, must be **greater** than m**R**, the maximum frictional force, which acts up the plane:

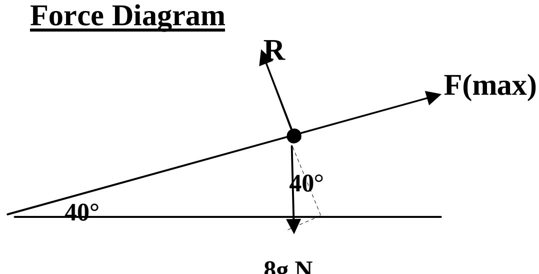
$\Rightarrow \quad mg \sin \theta > \mu\, mg \cos \theta$

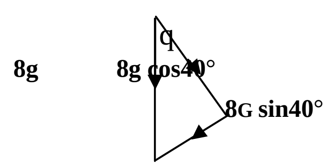
$\Rightarrow \quad \dfrac{\mathbf{mg\,sin}\ J}{\mathbf{mg\,cos}\ J} > m.$

$\therefore \quad$ **tan** q > m..

Example:
An object of mass **8 kg** rests on a rough plane surface inclined at 40° to the horizontal. Find the coefficient of friction between the object and the plane.

<u>**Force Diagram**</u> <u>**Vector Diagram**</u>





In the position of **limiting equilibrium**: **F** **=** m**R**

(Since the object is on the point of moving down the plane, the frictional force **F** is acting up the plane at its maximum value, μR.)

Resolving parallel to the plane: $8g \sin 40° = \mu R$

Resolving perpendicular to the plane: $8g \cos 40° = R \Rightarrow 8g \sin 40° = 8\mu g \cos 40°$

$$\Rightarrow \quad \frac{8g\sin 40^{o}}{8g\cos 40^{o}} = m$$

$$\therefore \quad \tan 40° = m$$

$$\Rightarrow \quad m = \mathbf{0.839}.$$

Motion up the plane

If the applied force is greater than the sum of the forces acting down the plane, motion will be upwards along the plane. In this case, the downward forces are **F(max)** and the **parallel component** of **mg**.

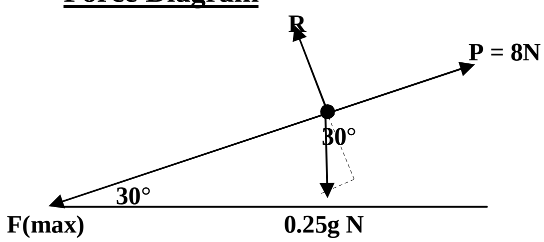
Example:

A body of mass **0.25kg** rests on a rough plane, inclined at an angle of **30°** to the horizontal.

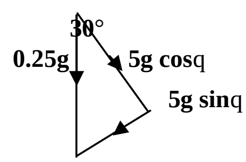
The coefficient of friction between the body and the plane is **0.2**.

Find the **acceleration** of the body when a force of **8N** is applied **up** the plane.

Force Diagram



Vector Diagram



Resolving perpendicular to the plane:				R = 0.25g cos 30°	=	2.12
F(max)=	μR	$\Rightarrow$	F(max)=	0.2×2.12	=	0.42N
Resolving parallel to the plane:				0.25g sin 30°	=	1.23

$\Rightarrow$ the component of the weight acting down the plane is 1.23N

Therefore the **forces acting down the plane** are 0.42N and 1.23N.

The force of **8N up the plane** must **exceed** the **sum** of the forces **down the plane**:

$$\Rightarrow \quad 8 - (0.42 + 1.23) = 0.25 \times \text{acceleration}$$

$$\Rightarrow \quad \text{acceleration} = 25.4\text{m/s}^2$$

MOMENTS

A moment is the **turning** effect of a force applied at a **point**, causing **rotational motion**. The moment of a force about a point is:

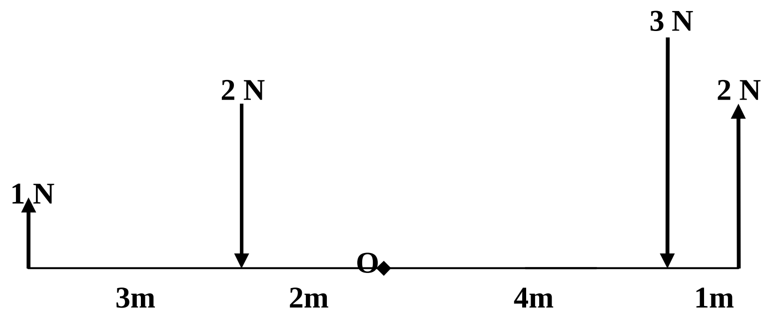
magnitude of the force ´ perpendicular distance of the force from the pivot.

If the force is measured in newtons and the distance in metres,

the **moment** of the force is measured in newton metres, **N m**.

Moments can be **clockwise** or **anti-clockwise** and should always have their sense clearly stated; a moment has **magnitude** and **direction**.

When finding the **resultant moment** of two or more forces about a point, one direction is taken as **positive** and the other **negative**.



Example:

Find the resultant moment about the point **O** of the forces shown in the diagram above.

Method:

Taking clockwise as $+$ and $-$ as anti-clockwise, we have:

$1 \times 5 - 2 \times 2 + 3 \times 4 - 2 \times 5 \;=\; 5 - 4 + 12 - 10 \;=\; 3$ Nm

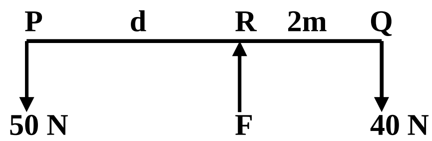
$\Rightarrow$ **3 N m clockwise**.

EQUILIBRIUM

For a body to be in equilibrium, the **resultant force** acting on the body must be **zero** and the **resultant moment** is **zero**.

Example 1:

A light horizontal beam **PQ** is in equilibrium. The beam carries masses of **5kg** and **4kg** at **P** and **Q** respectively. Find the values of **F** and **d**.
(Take g = 10m/s^2).



Since the beam is light, its mass is ignored.

F – 50 – 40 = 0 Þ F = 90N.

Taking moments around **R**, we have:

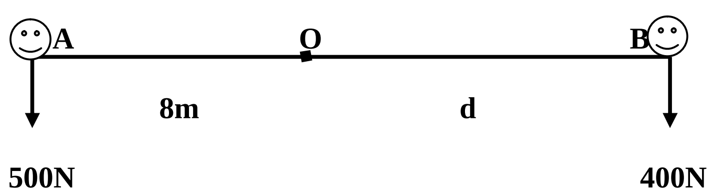
$$\begin{aligned} & 40 \times 2 - 50d &=& \; 0 \\ \Rightarrow \; & 80 - 50d &=& \; 0 \\ \Rightarrow \; & d &=& \; 1.6 \end{aligned}$$

$\therefore$ **d** is **1.6m** for the beam to be in equilibrium.

Example 2:

Two children, **A** and **B**, whose weights are **500N** and **400N** respectively, are balanced on a see-saw, whose pivot is at **O**.

If **A** is positioned **8m** from the pivot, how far from the pivot is **B**?



Taking moments around **O**, clockwise being **+**, we have:

$$\begin{aligned} & 400d - 500 \times 8 &=& \; 0 \\ \Rightarrow \; & 400d - 4000 &=& \; 0 \\ \Rightarrow \; & d &=& \; 10 \end{aligned}$$

$\therefore$ **B** is **10m** from the pivot for equilibrium.

KEY POINTS

Forces – Resolving; Resultant; Equilibrium; Friction; Moments.

- A **force** is a vector quantity that causes a **change** in the **state of motion** of a body. The **unit** of force is the **newton** (**N**).
  A force of 1**N** produces an acceleration of 1**m/s** ^{2} in a body of mass 1**kg**.
  The **weight** of a body is the force exerted upon it by **gravity** ($g = 9.81$ m/s ^{2}).

- Forces can be **resolved** into **two components** at **right-angles** to each other when a **right-angled triangle** is constructed around the force, making the **force** the **hypotenuse**. The forces are represented in **magnitude** and **direction** by the sides of the triangle in each case, using **addition of vectors**.

- **Two or more forces** acting at a point have the same effect as a **single force**, found by **vector addition**.
  This single force is called the **resultant** of the forces.

- If a body is **not moving**, then the **resultant force** in any direction must be **zero**. Hence, if **R** is the **resultant** of three forces **P**, **N** and **M**, then **–R** added to **P**, **N** and **M** will produce equilibrium.

- If a body of mass **M** kg rests on a horizontal surface, and a horizontal force of **P** N is applied to the body, **equal** and **opposite** forces act on the body and on the plane at **right-angles** to the surfaces in contact. *(Newton's Third Law).*
  The force, **F**, *opposing* the motion of the body, is called the **frictional force**.
  The frictional force acts in the **opposite** direction to the motion and is **parallel** to the surfaces that are in contact.
  The magnitude of **F(max)** is a **fraction** of the normal reaction **R**.
  This **fraction** is called the **coefficient** of **friction** and is denoted by m for the two surfaces in contact.
  We have: F(max)= μR.

- A **moment** is the turning effect of a force applied at a point, causing **rotational motion**.

  The moment of a force about a point is:

  magnitude of the force ´ perpendicular distance of the force from the pivot.

  If the force is measured in newtons and the distance in metres,

  the **moment** of the force is measured in newton metres, **N m**.

- For a body to be in **equilibrium**, the **resultant force** acting on the body must be **zero** and the **resultant moment** is **zero**.

NEWTON'S LAWS OF MOTION

First Law

A **CHANGE** IN THE **STATE OF MOTION** OF A BODY IS CAUSED BY A **FORCE.**

This means that a body will stay in a state of **rest** or in **constant motion** *unless* it is acted upon by an outside force.
If **forces** act on a body and it remains at **rest**, the forces must balance; hence the **resultant** force in any direction must be **zero**.
A body in motion can change its velocity or direction ***only if*** a resultant force acts upon it.
A **force**, therefore, is a **vector quantity** that causes a **change** in the **state of motion** of a body.

The **unit** of force is the **newton** (**N**).

A force of 1**N** produces an acceleration of 1**m/s^2** in a body of mass 1**kg**.
The **weight** of a body is the force exerted upon it by **gravity** (g = 9.81 m/s^2).

Generally, the **weight** of a body of **mass m kg** is **mg N**.

E.g. A person with a ***mass*** of **60 kg** has a ***weight*** of approximately **600 N** (g is often taken as 10 m/s^2).

Second Law

IF A **RESULTANT FORCE** ACTS ON A BODY, IT CAUSES **ACCELERATION.**

The acceleration is **proportional** to the **force** and the same force will **not** cause the same acceleration in all bodies; the acceleration depends on the **mass** of the body on which the force acts.
The standard **unit** of **mass** is the **kilogram** (kg).
A **force** of **1N** produces an **acceleration** of **1m/s^2** in a **mass** of **1kg**.

Generally, a force of **F** newtons acting on a body of mass **m** kg produces an acceleration of **a** m/s^2, giving the **equation** of **motion**:

$$\mathbf{F} = \mathbf{ma}.$$

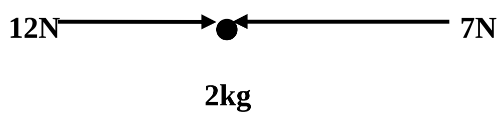
E.g. 1. A **force** of **5N** acting on a body of **mass 10kg** has an **acceleration** of **0.5m/s^2**:

$$\mathbf{5} = \mathbf{10a}$$
$$\Rightarrow \mathbf{a} = \mathbf{0.5}.$$

E.g. 2. The **resultant force** that would give a body of **mass 250g** an **acceleration** of **12m/s** ^{2} is **3N**:

$$\mathbf{F} = \mathbf{0.25(12)}$$
$$\Rightarrow \mathbf{F} = \mathbf{3N}.$$

E.g. 3. A body of **mass 2kg** rests on a smooth horizontal surface. Horizontal forces of **12N** and **7N** start to act on the particle in opposite directions.
Find the **acceleration** of the body.



$$\mathbf{F} = \mathbf{ma}$$
$$\Rightarrow \mathbf{12 - 7} = \mathbf{2a}$$
$$\Rightarrow \mathbf{5} = \mathbf{2a}$$
$$\Rightarrow \mathbf{2.5} = \mathbf{a}$$

$\Rightarrow$ acceleration of **2.5m/s** ^{2} .

VECTORS: APPLICATION OF F = MA

F = MA CAN BE APPLIED EASILY USING VECTORS.

Using Vectors with Constant Acceleration Formulae to describe Motion

F = ma and the **Constant Acceleration Formulae** may be used to deal with motion in two dimensions if the law or formula is changed to its **vector equivalent**.

Since **mass** and **time** are the only **scalars**, we have:

Law / Formula	**Vector Form**
$f = ma$	$\mathbf{f} = m\mathbf{a}$
$v = u + at$	$\mathbf{v} = \mathbf{u} + \mathbf{a}t$
$s = ut + \frac{1}{2}at^2$	$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$
$v^2 = u^2 + 2as$	$\mathbf{v.v} = \mathbf{u.u} + 2\mathbf{a.s}$

Example 1:

A particle of mass **2kg** is initially at rest at the point **P**, whose displacement from the origin **O** is **OP** = (**i** + 3**j**) metres. If the force **F** = 4**i** + 3**j** acts on the particle, find the displacement **OR** from **O** after 3 seconds.

Solution 1:

$\mathbf{F} = 4\mathbf{i} + 3\mathbf{j}$, $m = 2$:

$\mathbf{f} = m\mathbf{a}$ Þ $4\mathbf{i} + 3\mathbf{j} = 2\mathbf{a}$

Þ $2\mathbf{i} + 1.5\mathbf{j} = \mathbf{a}$

$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$:

Þ $\mathbf{s} = (0\mathbf{i} + 0\mathbf{j})3 + \frac{1}{2}(2\mathbf{i} + 1.5\mathbf{j})9$

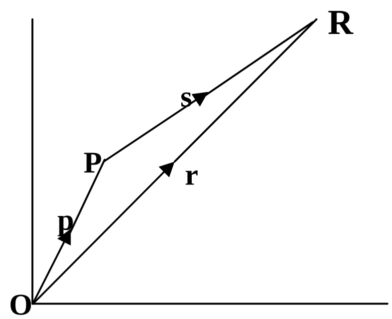
Þ $\mathbf{s} = 0 + 9\mathbf{i} + 6.75\mathbf{j}$

Þ $\mathbf{s} = 9\mathbf{i} + 6.75\mathbf{j}$

$\mathbf{r} = \mathbf{p} + \mathbf{s}$

Þ $\mathbf{r} = \mathbf{i} + 3\mathbf{j} + 9\mathbf{i} + 6.75\mathbf{j}$

Þ $\mathbf{r} = 10\mathbf{i} + 9.75\mathbf{j}$.



Example 2:

A particle of mass 6kg starts with velocity 3**i** + **j** at the point **P**, whose displacement from the origin **O** is **OP** = (**i** + 2**j**) metres. If the force **F** = 12**i** + 24**j** is acting on the particle, and after 3 seconds it has arrived at **R**, find:

(i) the velocity at **R**
(ii) the displacement **OR** from **O**.

Solution 2:

F = 12**i** + 24**j**, m = 6:

f = m**a**:
Þ $12\mathbf{i} + 24\mathbf{j} = 6\mathbf{a}$
Þ $2\mathbf{i} + 4\mathbf{j} = \mathbf{a}$

v = **u** + **a**t:
Þ $\mathbf{v} = 3\mathbf{i} + \mathbf{j} + (2\mathbf{i} + 4\mathbf{j})3$
Þ $\mathbf{v} = 3\mathbf{i} + \mathbf{j} + 6\mathbf{i} + 12\mathbf{j}$
Þ $\mathbf{v} = 9\mathbf{i} + 13\mathbf{j}$ (**a**)

$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$:

Þ $\mathbf{s} = (3\mathbf{i} + \mathbf{j})3 + \frac{1}{2}(2\mathbf{i} + 4\mathbf{j})9$
Þ $\mathbf{s} = 9\mathbf{i} + 3\mathbf{j} + 9\mathbf{i} + 18\mathbf{j}$
Þ $\mathbf{s} = 18\mathbf{i} + 21\mathbf{j}$

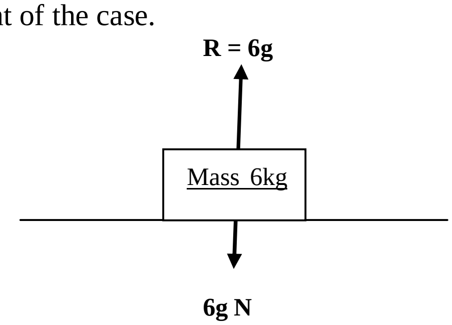
$\mathbf{r} = \mathbf{p} + \mathbf{s}$
Þ $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 18\mathbf{i} + 21\mathbf{j}$
Þ $\mathbf{r} = 19\mathbf{i} + 23\mathbf{j}$ (**b**)

Third Law

ACTION AND **REACTION** ARE **EQUAL** AND **OPPOSITE.**

This means that if two bodies **P** and **Q** are in contact and exert **forces** on each other, the **forces** are **equal** in **magnitude** and **opposite** in **direction**.

E.g. A case with a **mass** of **6kg** rests on a horizontal table. The case exerts a force on the table and the table 'reacts' by exerting an equal and opposite force on the case. Since the case is at rest, the reaction force **R** is **6g**, i.e. the weight of the case.



<u>CONNECTED PARTICLES</u>

In these problems, the **strings** connecting two particles are considered to be **light** and **inextensible**:

> Since the string is *light*, its **weight** can be **ignored**.
> Also, since it is *inextensible*, both particles have the **same speed** and **acceleration** while the string is kept *taut*.

By Newton's Third Law, the **tension** in the **string** acting on **both particles** is **equal in magnitude** and **opposite in direction**.
Note also that a *smooth* surface offers **no resistance** to the motion of a body across it.

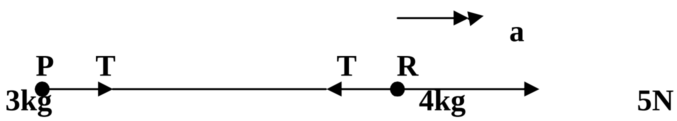
Example 1:

Two particles, **P** and **R** are at rest on a smooth horizontal surface.
P has mass **3kg** and **R** has mass **4kg**.
P is connected to **R** by a light inextensible string, which is taut.

A force of **5N** is exerted on particle **R** in the direction **PR**.
Find the acceleration of the two particles and the tension in the string.

Method:

The **diagram** looks like this:



Using **F = ma**:

Particle **P**:	**T**	=	**3a**	… (i)
Particle **R**:	**5 – T**	=	**4a**	… (ii)
(i) + (ii)	**5**	=	**7a**	
\	$\frac{5}{7}$	=	**a**.	
Substitute in (i):	**T**	=	$2\frac{1}{7}$.	

∴ the **acceleration** of the particles is $\frac{5}{7}$ **m/s**^{2}

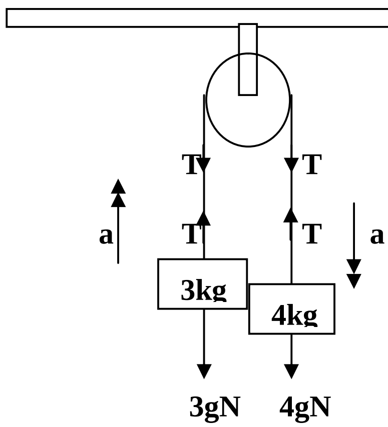
and the **tension** in the string is $2\frac{1}{7}$ **N**.

Example 2:
Two particles, **P** and **R** are connected by a light inextensible string passing over a smooth fixed pulley.
P has mass **3kg** and **R** has mass **4kg**.

The particles hang freely and are released from rest.
Find the acceleration of the two particles and the tension in the string.
(Take g = 9.8.)

Method:

The **diagram** looks like this:



Using **F = ma**:

Particle **P**:	**T – 3g**	=	**3a**	… (i)
Particle **R**:	**4g – T**	=	**4a**	… (ii)
(i) + (ii)	**g**	=	**7a**	
\	$\frac{1}{7}$**g**	=	**a**.	

Substitute in (i): **T** = **3g +** $\frac{3}{7}$**g**.

∴ the **acceleration** of the particles is **1.4 m/s** ^{2}

and the **tension** in the string is **33.6 N**.

KEY POINTS

Newton's Laws of Motion – Connected Particles; Resultant : F = ma

• NEWTON'S LAWS OF MOTION

First Law: A **change** in the **state of motion** of a body is caused by a **force**.

Second Law: If a **resultant force** acts on a body, it causes **acceleration**.
The **equation of motion** is: **F = ma**.
F = ma and the **Constant Acceleration Formulae** may be used to deal with motion in two dimensions if the law or formula is changed to its **vector equivalent**.

Since **mass** and **time** are the only **scalars**, we have:

Law / Formula	**Vector Form**
$f = ma$	$\mathbf{f} = m\mathbf{a}$
$v = u + at$	$\mathbf{v} = \mathbf{u} + \mathbf{a}t$
$s = ut + \frac{1}{2}at^2$	$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$
$v^2 = u^2 + 2as$	$\mathbf{v}.\mathbf{v} = \mathbf{u}.\mathbf{u} + 2\mathbf{a}.\mathbf{s}$

Third Law: **Action** and **reaction** are **equal** and **opposite**.

This means that if two bodies **P** and **Q** are in contact and exert **forces** on each other, the **forces** are **equal** in **magnitude** and **opposite** in **direction**.

• CONNECTED PARTICLES

In these problems, the **strings** connecting two particles are considered to be **light** and **inextensible**:

Since the string is *light*, its **weight** can be **ignored**.
Also, since it is *inextensible*, both particles have the **same speed** and **acceleration** while the string is kept *taut*.

By Newton's Third Law, the **tension** in the **string** acting on **both particles** is **equal in magnitude** and **opposite in direction**.

Note also that a *smooth* surface offers **no resistance** to the motion of a body across it.

EXAMINATION – STYLE QUESTIONS

C.C.E.A. ADDITIONAL MATHEMATICS PAST PAPER 2 QUESTIONS

1. (Throughout this question **i** and **j** denote unit vectors parallel to a set of standard x - y axes.)

   Three vectors **a**, **b** and **c** are defined as follows:
   a = i + 2j **b = -3i + j** **c = 9i + 4j**

   Calculate the magnitude and direction of the single vector represented by **4a – 2b – c**, giving your answers correct to 1 decimal place.

2. (Throughout this question **i** and **j** denote unit vectors parallel to a set of standard x - y axes.)

   A package rests in equilibrium on a smooth horizontal plane under the action of three horizontal forces:
   $\mathbf{F}_1 = (\mathbf{-3i + 4j})$ **N** $\mathbf{F}_2 = (\mathbf{4i - 7j})$ **N** $\mathbf{F}_3 = (x\mathbf{i} + y\mathbf{j})$ **N**.

   (i) Find the values of ***x*** and ***j***.

   The force $\mathbf{F}_3$ is now removed and the package begins to move under the action of $\mathbf{F}_1$ and $\mathbf{F}_2$ only.

   Find
   (ii) the resultant force **R** acting on the package as a vector in terms of the unit vectors **i** and **j**.

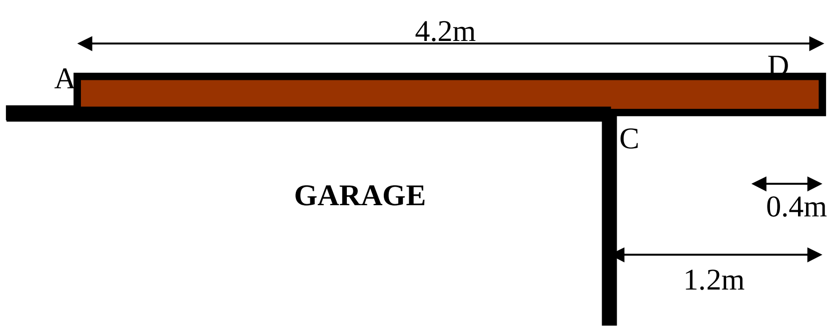
   (iii) the magnitude of **R**, giving your answer correct to 2 decimal places.

3. A particle of mass 4kg moves in a horizontal plane under the action of a constant horizontal force. The initial velocity of the particle is (**5i – j**) m/s and 2 seconds later its velocity is (**8i + 3j**) m/s.

   Find:

   (i) the acceleration of the particle;
   (ii) the magnitude of the horizontal force acting on the particle;
   (iii) the angle which this force makes with the vector **i**, giving your answer in degrees correct to 1 decimal place.

4. A uniform plank AB of length 4.2m lies on the horizontal roof of a garage. The plank projects over the edge C of the roof as shown in the figure below:



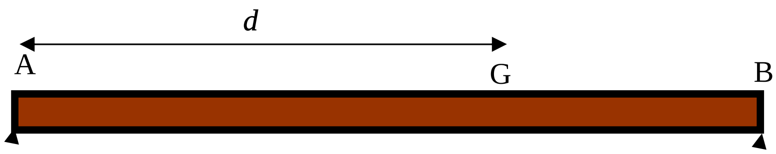
A boy of mass 63kg walks along the plank from the end A. When he reaches a point D, 0.4m from the end B, the plank is on the point of tilting about the edge C.

(i) State the reaction at A when the plank is on the point of tilting.
(ii) Find the mass of the plank.

A mass *M* kg is placed on the plank at A. The boy now walks along the plank towards the end B. When he reaches B the plank is on the point of tilting about the edge C.

(iii) By taking moments about C, find the value of *M*.
(iv) Calculate the magnitude of the force exerted on the plank at C.

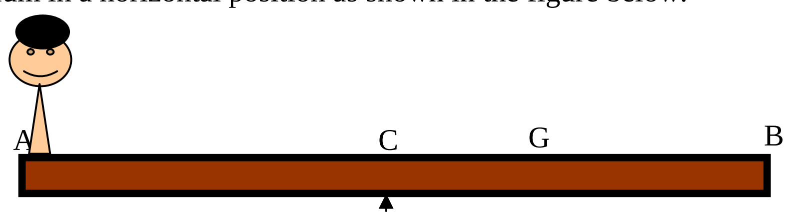
5. A **non-uniform** plank AB of length 8 metres and mass *M* kg rests in equilibrium on supports at each end of the plank. The reactions at A and B are 15g N and 35g N respectively. The centre of mass of the plank is at G where AG = *d* metres as shown in the figure below.



Calculate

(i) the mass *M* of the plank,
(ii) the distance *d* of the centre of mass from A.

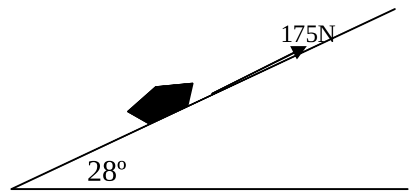
One of the supports is now removed and the other is placed at the centre C of the plank. A child of mass *m* kg stands on the plank at A and the plank again rests in equilibrium in a horizontal position as shown in the figure below.



(iii) Find the mass *m* of the child.

6. **Throughout this question take g = 10m/s^2**

A boat of mass 200kg stands on a rough ramp inclined at 28º to the horizontal.
A light inextensible rope attached to the rope prevents it from sliding down the ramp as shown in the figure below:



When the tension in the rope is 175N the boat is on the point of sliding down the ramp.

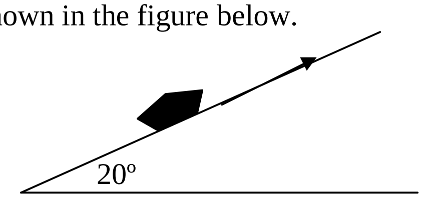
(i) Copy the diagram overleaf and include on it **all** the forces acting on the boat.
(ii) Calculate the coefficient of friction between the boat and the ramp, giving your answer correct to two decimal places.

The rope is removed and the boat moves down the ramp from rest. The coefficient of friction between the boat and the ramp remains the same.

(iii) Calculate the acceleration of the boat down the ramp.

7. **Throughout this question take g = 10m/s^2**

A sleigh of mass 20kg is initially at rest on a rough slope inclined at 20º to the horizontal as shown in the figure below.



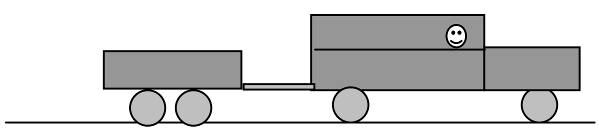
The sleigh is released from rest, and moves down the slope. After 10 seconds it is moving with a velocity of 8 m/s.

(i) Copy the diagram and include on it **all** the forces acting on the sleigh.

Find

(ii) the acceleration of the sleigh,
(iii) the frictional force opposing the motion of the sleigh, giving your answer correct to 2 decimal places,
(iv) the coefficient of friction between the sleigh and the rough surface, giving your answer correct to 2 decimal places.

8. A car of mass 1000kg pulls a trailer of mass 800kg by means of a light inextensible horizontal tow-bar along a straight road with an acceleration of 1.5m/s^2 as shown in the figure below:



The car's engine exerts a tractive force of 4000N and the resistance to motion of the car is 0.8N per kg.

Find
(i) the resistance to motion of the trailer;
(ii) the tension in the tow-bar.

At the instant when the speed of the car and the trailer is 10m/s the tow-bar breaks. Find
(iii) the time taken by the trailer in coming to rest;
(iv) the distance it travels in this time.

9. A car of mass 1200kg is towing a trailer of mass 600kg by means of a light horizontal tow-bar. The car and trailer are travelling along a straight horizontal road at a constant speed of 12 m/s. The resistance to the motion of the car is 1.2 N per kg of mass. The resistance to the motion of the trailer is 0.8 N per kg of mass.

Find
(i) the tractive force of the car's engine,
(ii) the tension in the tow-bar.

After travelling for a short time at 12 m/s the car accelerates at 0.3 m/s^2.

Find
(iii) the new tractive force of the engine,
(iv) the speed of the car and trailer after accelerating for a period of 25 seconds.

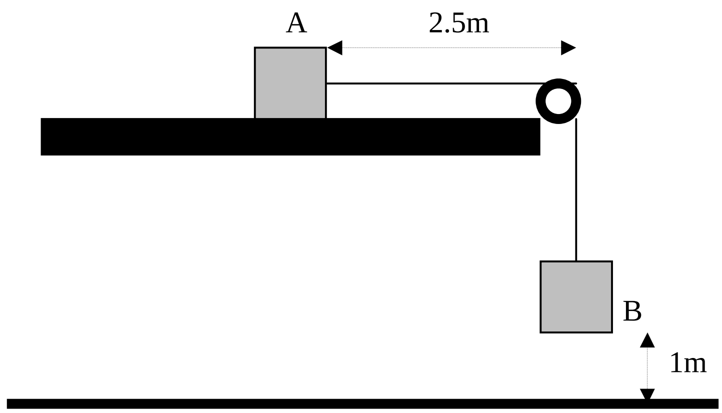
At the end of this period the tow-bar breaks.

Find, giving your answer to the nearest metre,

(v) the additional distance travelled by the trailer before it comes to rest.

10. **Throughout this question take g = 10m/s^2**

Two packages A and B of masses 3kg and 2kg respectively are connected by a light inextensible string which passes over a smooth pulley fixed at the edge of a smooth horizontal platform. Package A is held on the platform 2.5m from the pulley. Package B hangs freely 1m above the ground as shown in the figure below:



Package A is released from rest.

Calculate:

(i) the acceleration of the packages;
(ii) the tension in the string;
(iii) the speed with which B hits the ground, giving your answer correct to 2 decimal places;
(iv) the time which elapses between B hitting the ground and A reaching the pulley, giving your answer correct to 2 decimal places.

11. **Throughout this question take g = 10m/s^2**

A child throws a ball vertically upwards. The ball leaves the child's hand, which is 1 metre above the ground, with a speed of 12m/s.

(i) Show that the greatest height, **above the ground**, reached by the ball is 8.2m.
(ii) Find the speed with which the ball hits the ground, giving your answer correct to 1 decimal place.
(iii) Determine the time (in seconds) that elapses from the instant the child throws the ball until the ball hits the ground, giving your answer correct to 1 decimal place.

12. A particle P moves in a straight line and passes through a fixed point O. At time t seconds after passing through the point O its velocity v m/s is given by:

$$v = 16 + 6t - t^2 .$$

Find:

(i) the initial velocity of P;
(ii) the time when the velocity is zero;
(iii) the acceleration of P when its velocity is zero;
(iv) the maximum velocity reached by the particle.

13. A motorist travelling on a straight stretch of motorway at a steady speed of 32m/s sees a warning sign for roadworks asking motorists to reduce speed. She immediately decelerates uniformly for 24 seconds until she reaches a speed of 20m/s. She travels at this speed for 1500m and then accelerates at 1.2m/s^2 until she regains her original speed of 32m/s.

(i) Draw a speed/time graph to illustrate this information, marking the time intervals on the horizontal axis.
(ii) Calculate how far, in metres, the motorist travels from the time she sees the warning sign until she regains her original speed.
(iii) Calculate how much less time it would have taken the motorist to complete her journey if there had been no roadworks.

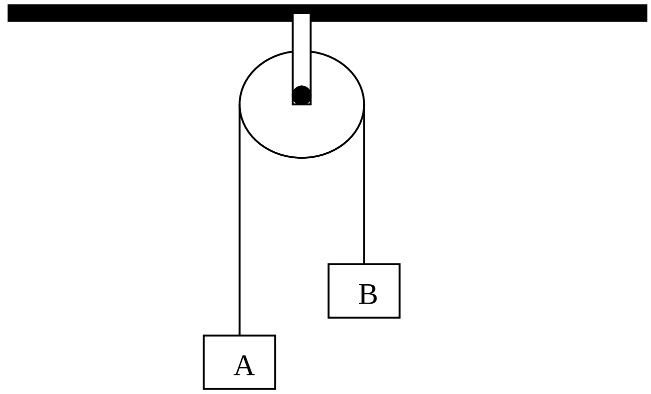
14. An athlete running on a straight horizontal track passes a marker at a constant speed of 3m/s. Two minutes later a cyclist starts off from the same marker and accelerates from rest at 0.3m/s^2 for 30 seconds until he reaches his maximum speed. He maintains this speed until he overtakes the athlete. He immediately decelerates uniformly and comes to rest 45 seconds after passing the athlete.

Find:

(i) the maximum speed of the cyclist;
(ii) the distance from the marker at which the cyclist overtakes the athlete;
(iii) the deceleration of the cyclist;
(iv) the distance between the athlete and the cyclist when the cyclist comes to rest.

15. **Throughout this question take g = 10m/s^2**

Two packages, A and B, of masses 15kg and 9kg respectively are connected by a light inextensible string passing over a smooth fixed pulley as shown in the figure below:

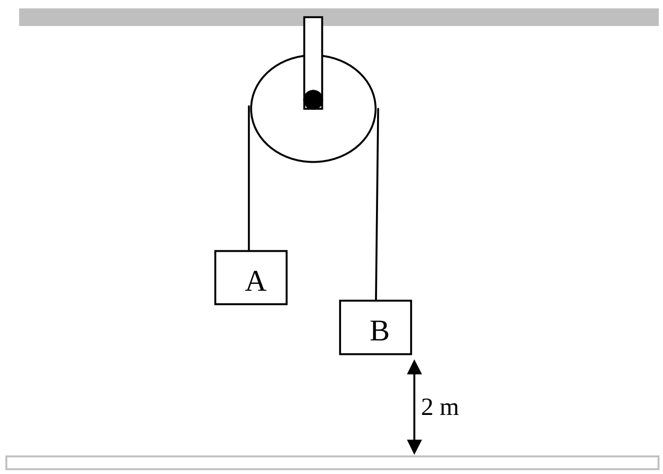


The packages are released from rest with the string taut.

Find:
(i) the acceleration of the packages;
(ii) the tension in the string;
(iii) the force exerted by the string by the pulley.

16. **Throughout this question take g = 10m/s^2**

Two packages, A and B, of masses 3kg and 5kg respectively are connected by a light inextensible rope passing over a smooth pulley fixed to the ceiling of a warehouse as shown in the figure below:



The packages are held so that both parts of the rope are vertical and B is 2 metres above the floor of the warehouse. The packages are then released from rest with the rope taut.

(i) Copy the figure and mark on your diagram all the forces acting on the pulley and on the packages.

Assuming that A does not reach the pulley in the ensuing motion, calculate

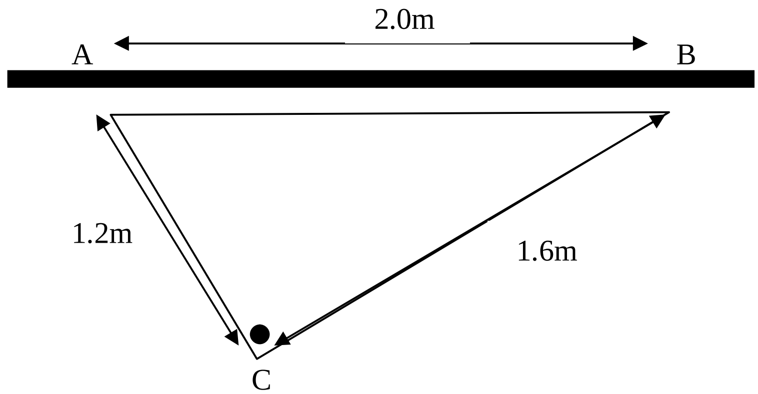
(ii) the acceleration of the packages,
(iii) the force on the pulley when the packages are in motion,
(iv) the speed of B when it hits the ground, giving your answer to 1 decimal place.

When B hits the ground the rope becomes slack and A initially continues to rise.

(v) Calculate the **additional** distance through which A rises after B hits the ground.

17. (**Take g = 10m/s** ^{2}.)

A hazard light of mass 2kg is held in equilibrium at a point C by two wires CA and CB. The lengths of CA and CB are 1.2m and 1.6m respectively. The points A and B are fixed 2.0m apart on a horizontal bar, as shown in the figure below:



(i) Show that the angle ACB is a right angle.

Calculate:

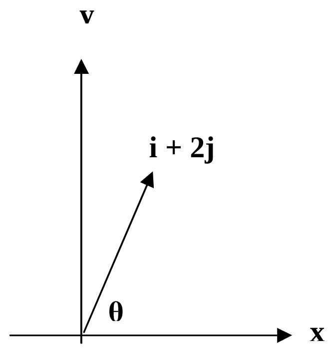
(ii) the tension T_1 in the wire CA;

(iii) the tension T_2 in the wire CB.

1. **a** = **i** + 2**j**; **b** = -3**i** + **j**; **c** = 9**i** + 4**j**.

$$\mathbf{4a - 2b - c = 4(i + 2j) - 2(-3i + j) - (9i + 4j) \quad = \quad i + 2j}.$$

$$|\mathbf{i + 2j}| \quad = \quad \sqrt{1^2 + 2^2} \quad = \quad \sqrt{5} \quad = \quad 2.2.$$



$$\Theta \quad = \quad \tan^{-1}\left(\frac{2}{1}\right) \quad = \quad 63.4^\circ.$$

2. (i)

$$\mathbf{F_1 + F_2 + F_3} \quad = \quad \mathbf{0}$$

$$\mathbf{-3i + 4j + 4i - 7j + xi + yj} \quad = \quad \mathbf{0i + 0j}$$

$$\Rightarrow \quad \mathbf{(-3 + 4 + x)i + (4 - 7 + y)j} \quad = \quad \mathbf{0i + 0j}.$$

Equating coefficients gives:

$1 + x = 0$ and $-3 + y = 0$

$\Rightarrow$ **x = -1 and y = 3**.

(ii)

$$\mathbf{R} \quad = \quad \mathbf{F_1 + F_2}$$

$$\Rightarrow \quad \mathbf{R} \quad = \quad \mathbf{-3i + 4j + 4i - 7j}$$

$$\Rightarrow \quad \mathbf{R} \quad = \quad \mathbf{(i - 3j)N}.$$

(iii)

$$|\mathbf{R}| \quad = \quad \sqrt{1^2 + (-3)^2} \quad = \quad \sqrt{1 + 9} = \quad \sqrt{10}$$

$$\Rightarrow \quad |\mathbf{R}| \quad = \quad \mathbf{3.16)N}.$$

3. (i) $\mathbf{v} = \boldsymbol{u} + \boldsymbol{a}\mathbf{t}$;

$\mathbf{8i + 3j} = \mathbf{5i - j + 2}\boldsymbol{a}$

$\mathbf{2}\boldsymbol{a} = \mathbf{3i + 4j}$

$\boldsymbol{a} = (\frac{\mathbf{3}}{\mathbf{2}}\mathbf{i + 2j})\mathbf{m/s}^{\mathbf{2}}$.

(ii) $\mathbf{F} = \mathbf{m}\boldsymbol{a}$;

$\mathbf{F} = \mathbf{4}(\frac{\mathbf{3}}{\mathbf{2}}\mathbf{i + 2j})$

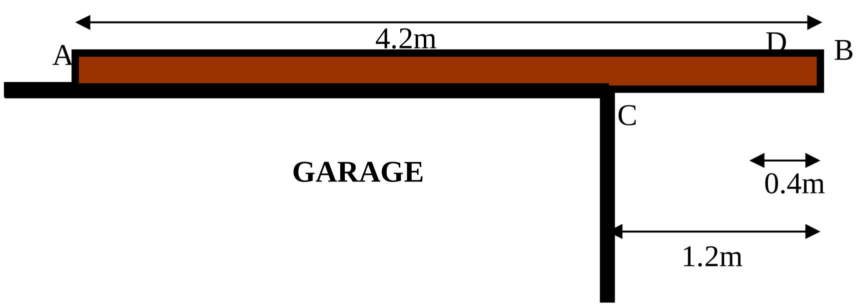
$\mathbf{F} = \mathbf{(6i + 8j)N}$

$|\mathbf{F}| = \sqrt{\mathbf{6^2 + 8^2}} = \mathbf{10N}$.

(iii) $\mathbf{Tan}\ \boldsymbol{\theta} = \frac{\mathbf{8}}{\mathbf{6}}$

$\boldsymbol{\theta} = \mathbf{tan}^{-1}\left(\frac{\mathbf{8}}{\mathbf{6}}\right) = \mathbf{53.1^\circ}$.

4.



(i) When the plank is at the point of tilting, the reaction at **A** is **zero**.

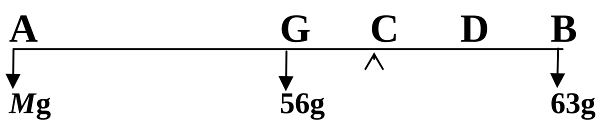
(ii) Taking moments about **C**:

$\text{Mg} \times 0.9 = 63\text{g} \times 0.8$

$\Rightarrow \quad \mathbf{M} = \frac{\mathbf{63g \times 0.8}}{\mathbf{0.9}}$

$\therefore \quad \mathbf{M} = \mathbf{56kg}$.

(iii) Let the mass at **A** be ***M***kg.



Taking moments about **C**:

$\text{Mg} \times \text{AC} + 56\text{g} \times \text{GC} = 63\text{g} \times \text{BC}$

Þ $Mg \times 3.0 + 56g \times 0.9 = 63g \times 1.2$

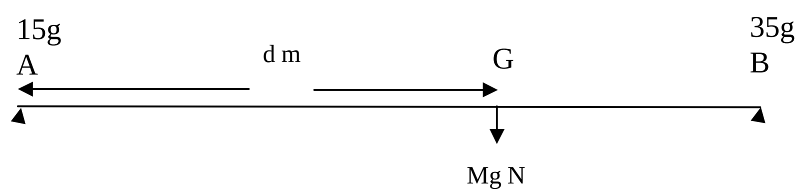
Þ $M = \dfrac{(63g ´ 1.2) - (56g ´ 0.9)}{3.0}$

\ $M = 8.4\text{kg}$.

(iv) The force exerted on the plank at **C** is **total** of the forces **downwards**:

Þ **84 + 560 + 630 = 1274N**.

5.



(i) The plank is in equilibrium if the forces acting upwards balance those acting downwards:

$\Rightarrow$ Mg = 15g + 35g

$\Rightarrow$ Mg = 50g

$\Rightarrow$ M = 50kg.

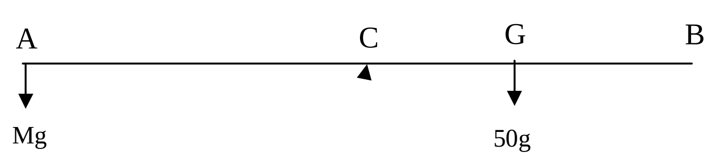
(ii) Take moments at A :

50g ´ d = 35g ´ 8

$\mathbf{d} = \dfrac{\mathbf{35g ´ 8}}{\mathbf{50g}}$

Þ **d = 5.6 m**.

(iii)



Take moments at C :

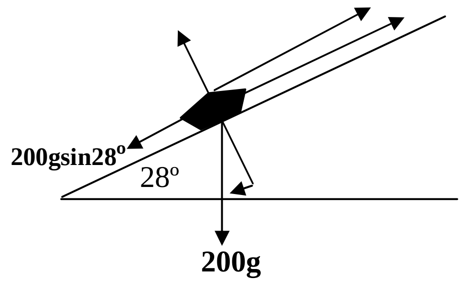
mg ´ 4 = 50g ´ (5.6 – 4)

4mg = 50g ´ 1.6

Þ $\mathbf{m} = \dfrac{\mathbf{50g ´ 1.6}}{\mathbf{4g}}$

Þ **m = 20kg**.

6. (i)



(ii) $200g \sin 28^{\circ} = \mu R + 175.$

$R = 200g \cos 28^{\circ}$

$\Rightarrow \quad 200g \sin 28^{\circ} = \mu(200g \cos 28^{\circ}) + 175.$

$$\therefore \quad \frac{200g \sin 28^{\circ} - 175}{200g \cos 28^{\circ}} = \mu$$

$\Rightarrow \quad \mu = 0.43.$

(iii) **Accelerating Force** $= 200g \sin 28^{\circ} - \mu R$

$\Rightarrow \quad 200g \sin 28^{\circ} - 0.43(200g \cos 28^{\circ})$

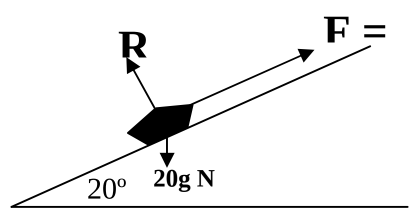
$\Rightarrow \quad 179.6\text{N}.$

Accelerating Force = **Mass × Acceleration**

$\Rightarrow \quad 179.6 = 200 \times a$

$\Rightarrow \quad a = 0.90 \text{ m/s}^2.$

7. (i)



(ii) Using $v = u + at$:

$8 = 0 + 10a$

$\Rightarrow \quad a = 0.8\text{m/s}^2.$

(iii) **Force acting down** the plane = $20g \sin 20^{\circ}$

$\Rightarrow$ **accelerating force** = $20g \sin 20^{\circ} - F$ (the frictional force).

Accelerating force = **mass × acceleration**

$\Rightarrow \quad 20g \sin 20^{\circ} - F = 20 \times 0.8$

Þ F = 20g sin 20º - 16

Þ F = 68.40 - 16

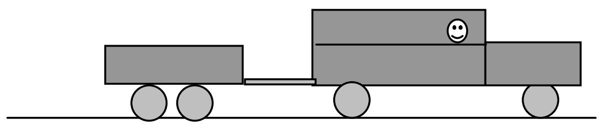
Þ F = 52.40 N.

(iv) R = 20g cos 20º = 187.94 N.

F = mR

Þ m = $\frac{52.40}{187.94}$ = 0.28 N.

8.



(i) Let **R** be the resistance to motion of the trailer ($\overleftarrow{\textbf{R}}$).

The resistance to motion of the car is **0.8N** per kg:
1000 × 0.8 = 800N (←).

Tractive force exerted by the engine = **4000N** (→).

F = ma:

Þ **F = 1800 × 1.5 = 2700N** (→).

\ **2700 = 4000 – 800 – R**

Þ **R = 4000 – 800 – 2700**.

\ **R = 500N**.

(ii) Let **T** be the tension in the tow-bar (→).

500N is the resistance to motion of the trailer (←).

F = ma:

Þ **T - 500 = 800 × 1.5**

Þ **T - 500 = 1200** (→)

\ **T = 1700N**.

(iii) When tow-bar breaks, **T = 0**; **500N** is the retarding force of the trailer.

F = ma:

Þ **500 = 800 × a**

$\Rightarrow$ $\quad \mathbf{a} = \mathbf{\frac{5}{8}\,m/s^2}$ retardation.

$\mathbf{v} = \mathbf{u + at}$:

$\Rightarrow$ $\quad \mathbf{0} = \mathbf{10 - \frac{5}{8}t}$

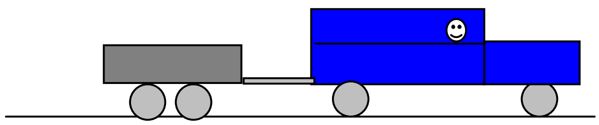
$\Rightarrow$ $\quad \mathbf{t} = \mathbf{16}$ seconds.

(iv) $\quad \mathbf{v^2} = \mathbf{u^2 + 2as}$:

$\Rightarrow$ $\quad \mathbf{0} = \mathbf{10^2 - 2 \times \frac{5}{8}s}$

$\Rightarrow$ $\quad \mathbf{s} = \mathbf{80}$ metres.

9.



(i) At constant speed **Tractive Force = Resistance**.

Total resistance (car + trailer) = **(1200 × 1.2) + (600 × 0.8)**.

$\Rightarrow$ **Tractive force = 1440 + 480 = 1920 N**.

(ii) **Tension in tow-bar = Resistance to motion of trailer**.

$\Rightarrow$ **T = 480 N**.

(iii) **New tractive force – Resistance = Accelerating force**

$\Rightarrow$ **New Tractive Force = Resistance + Accelerating Force**

Accelerating Force:

AF = ma:

$\Rightarrow$ **AF = 1800 × 0.3**

$\Rightarrow$ **AF = 540 N**.

$\Rightarrow$ **New tractive force = 1920 + 540 = 2460 N**.

(iv) $\quad \mathbf{v} = \mathbf{u + at}$:

$\Rightarrow$ $\quad \mathbf{v} = \mathbf{12 + (0.3 \times 25)} = \mathbf{19.5\,m/s}$.

(v) **Retarding** force = **Resistance to motion** of **trailer** = **480 N**.

$\mathbf{F} = \mathbf{ma}$:

Þ $\mathbf{480} = \mathbf{600a}$

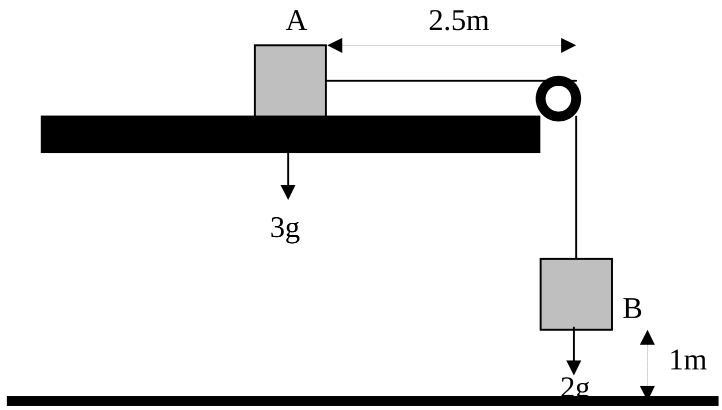
Þ $\mathbf{\frac{480}{600}} = \mathbf{0.8m/s^2} = \mathbf{a}$.

$\mathbf{v^2} = \mathbf{u^2 + 2as}$:

Þ $\mathbf{0} = \mathbf{19.5^2 + 2 \times (-0.8) \times s}$

Þ $\mathbf{s} = \mathbf{\frac{19.5^2}{1.6} = 237.6\ldots = 238\ m}$ (to nearest metre).

10.



(i) At **B:** $\mathbf{2g - T} = \mathbf{2a}$

At **A:** $\mathbf{T} = \mathbf{3a}$

$\mathbf{2g} = \mathbf{5a}$

Þ $\mathbf{20} = \mathbf{5a}$

Þ $\mathbf{a} = \mathbf{4m/s^2}$.

(ii) $\mathbf{T} = \mathbf{3a}$

Þ $\mathbf{T} = \mathbf{12N}$.

(iii) $\mathbf{v^2} = \mathbf{u^2 + 2as}$

Þ $\mathbf{v^2} = \mathbf{0 + 2(4)(1)} = \mathbf{8}$

Þ $\mathbf{v} = \mathbf{2.83m/s}$.

(iv) When **B** hits the ground, **A** is travelling at **2.83m/s** and still has **1.5m** to go before reaching the edge.

Þ $\dfrac{\mathbf{1.5}}{\mathbf{2.83}}$ = **0.53s**.

11. **Constant Acceleration Formulae**:

Treat the **child's hand** as the **origin** and **upwards** as + and **downwards** as -

$\mathbf{v = u + at}$; $\mathbf{v^2 = u^2 + 2as}$; $\mathbf{s = ut + ½at^2}$.

(i) $\mathbf{u = 12m/s}$; $\mathbf{a = -10m/s^2}$.

At greatest height, **v = 0**:

Using $\mathbf{v^2 = u^2 + 2as}$, we have:

$$\mathbf{0 = 12^2 + 2(-10)s}$$
$$\mathbf{20s = 144}$$
$$\mathbf{s} = \mathbf{7.2}\text{m above the child's hand.}$$
$$\mathbf{7.2 + 1} = \mathbf{8.2}\text{m above the ground.}$$

Q.E.D.

(ii) $\mathbf{v^2 = u^2 + 2as}$ (**s = -1**m ↓)

$$\mathbf{v^2 = 144 + 2(-10)(-1)}$$
$$\mathbf{v^2 = 164}$$
$$\mathbf{v = 12.8m}.$$

(iii)
$$\mathbf{v = u + at}$$
$$\mathbf{-12.806 = 12 - 10t} \quad (\mathbf{v = -12.806})$$
$$\mathbf{10t = 12 + 12.806}$$
$$\mathbf{t = 2.48}$$

Þ **2.5** seconds.

12. (i) $\mathbf{v = 16 + 6t - t^2}$

t = 0, **v** = **16m/s**.

(ii) **v = 0** Þ $\mathbf{16 + 6t - t^2 = 0}$

Þ $\mathbf{(8 - t)(2 + t) = 0}$

Þ $\mathbf{8 - t = 0}$

Þ $\mathbf{t = 8s}$.

(iii) $\quad \mathbf{a} = \dfrac{\mathbf{dv}}{\mathbf{dt}} = \mathbf{6 - 2t}$.

$\mathbf{t = 8}$, $\quad \mathbf{a} = \mathbf{6 - 16} = \mathbf{-10 m/s^2}$

Þ **deceleration** of $\mathbf{10 m/s^2}$.

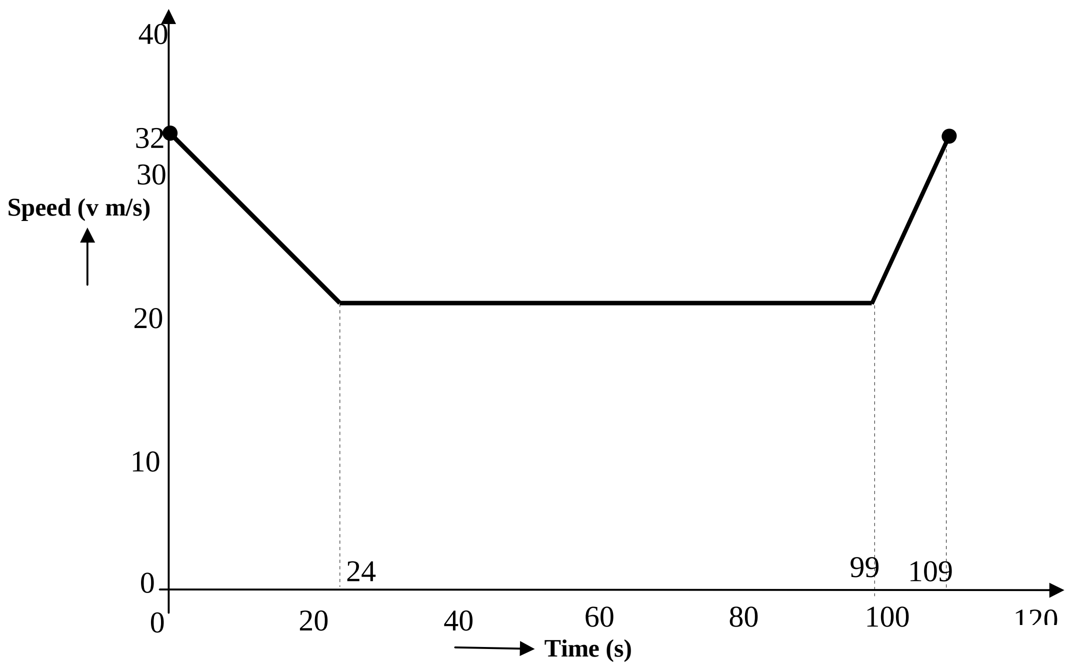
(iv) **Maximum velocity** Þ $\dfrac{\mathbf{dv}}{\mathbf{dt}} = \mathbf{0}$

Þ $\mathbf{6 - 2t} = \mathbf{0}$

Þ $\mathbf{t} = \mathbf{3}$.

$\mathbf{t = 3}$ Þ $\mathbf{16 + 6t - t^2} = \mathbf{25 m/s}$.

13.



(i) **Time** = $\dfrac{\mathbf{Distance}}{\mathbf{Speed}}$ Þ $\dfrac{\mathbf{1500}}{\mathbf{20}} =$ **75** seconds.

$\mathbf{v} = \mathbf{u + at}$

Þ $\mathbf{32} = \mathbf{20 + 1.2t}$

Þ $\mathbf{t} = \mathbf{10}$

Þ **10** seconds.

(ii) **Distance** = **Area under speed/time graph**:

Trapezium: $\frac{1}{2}$**(32 + 20)24** **=** **624**

Rectangle: **75 × 20** **=** **1500**

Trapezium: $\frac{1}{2}$**(32 + 20)24** **=** **624**

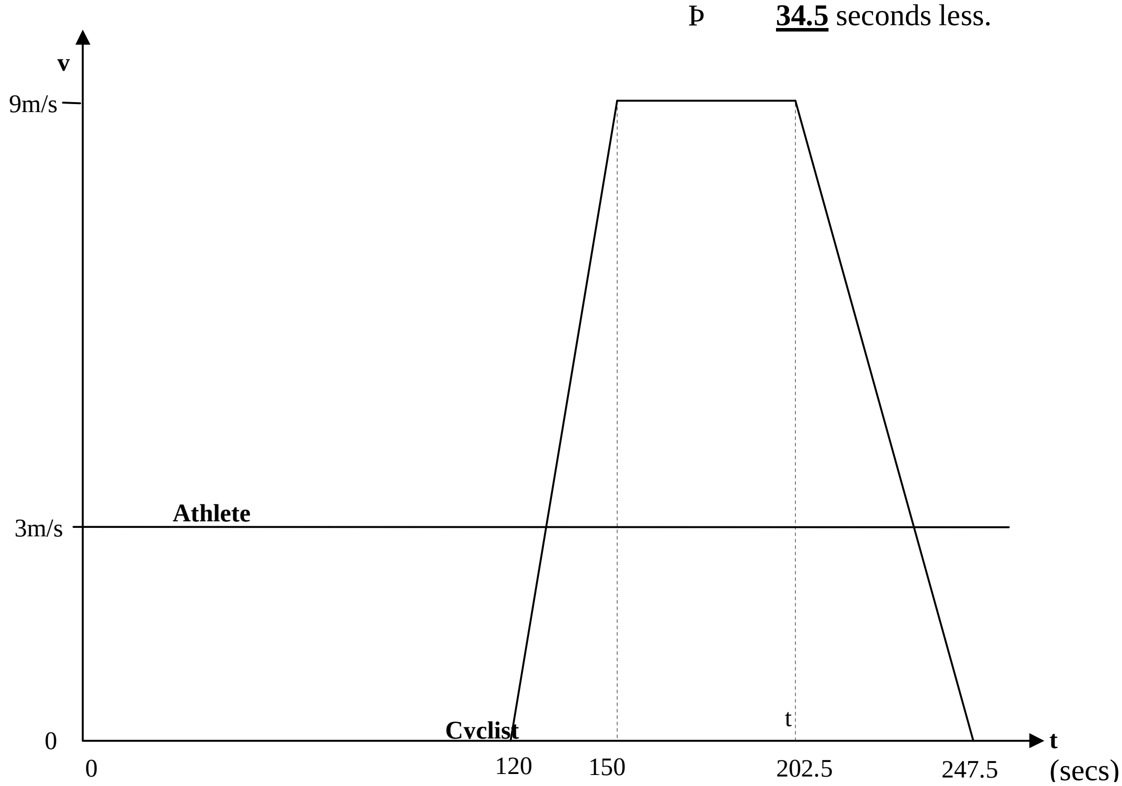
2384m.

(iii) **With roadworks**, **time**: **109** seconds.

Without roadworks, **time** = $\frac{\mathbf{2384}}{\mathbf{32}}$ = **74.5**: **74.5** seconds.

Þ **34.5** seconds less.

14.



(i) **Cyclist**: **Gradient** of line while accelerating = **0.3**

Þ $\frac{\mathbf{v}}{\mathbf{30}}$ **=** **0.3**

Þ **v** **=** **9m/s**.

(ii) Let cyclist overtake athlete **t** seconds after start:

Distance travelled by **athlete** to point of overtaking: **3t**.

Distance travelled by **cyclist** to point of overtaking:

Area of 1st triangle: $\frac{1}{2}(30 \times 9) = 135\text{m}$.

Area of rectangle: $9(t - 150) = (9t - 1350)\text{m}$.

Total Distance travelled by cyclist to **point of overtaking**:

$$135 + 9t - 1350 = 9t - 1215.$$
$$9t - 1215 = 3t$$
$$Þ \quad t = 202.5\text{s}$$
$$Þ \quad 3t = 607.5.$$
$$Þ \quad 607.5\text{m}.$$

(iii) **Cyclist**: **Gradient** of line while decelerating = $\frac{9}{45}$

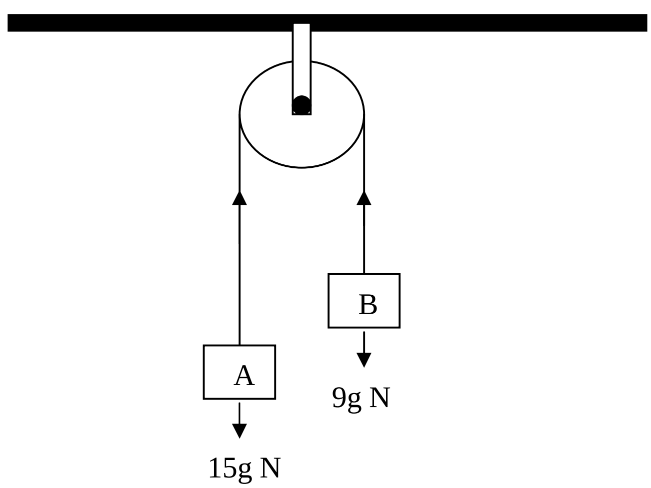
$$Þ \quad \textbf{deceleration} = 0.2\text{m/s}^2.$$

(iv) Distance travelled by cyclist in last 45 seconds: $\frac{1}{2}(45 \times 9) = 202.5\text{m}$.

Distance travelled by athlete in last 45 seconds: $45 \times 3 = 135\text{m}$.

Distance between them: $202.5 - 135 = 67.5\text{m}$.

15.



(i)
$$15g - T = 15a \quad \ldots \text{(i)}$$
$$-9g + T = 9a \quad \ldots \text{(ii)}$$
$$6g = 24a \quad \ldots \text{(i) + (ii)}$$

$$a = \frac{60}{24} = 2.4\text{m/s}^2.$$

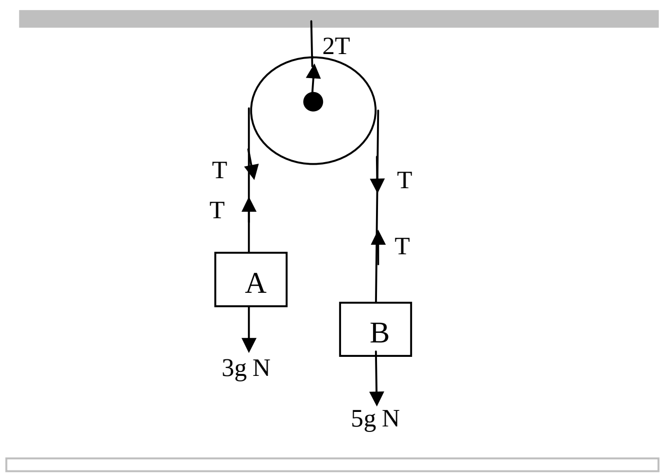
(ii) T = 9a + 9g

Þ **T = 22.5 + 90 = 112.5N**.

(iii) P = 2T

Þ **P = 2 × 112.5 = 225N**.

16. (i)



(ii) **5g – T = 5a** (i)

T - 3g = 3a (ii)

Eliminate T:

(i) + (ii) … **2g = 8a**

Þ **20 = 8a**

Þ **a = 2.5m/s^{2}**.

(iii) $T - 30 = 3 \times 2.5$

Þ $T = 7.5 + 30 = 37.5$ N

Þ $2T = 75$ N

Þ **Force on pulley** = **75 N** (Force on pulley is 2T)

(iv) $v^2 = u^2 + 2as$

Þ $v^2 = 0 + 2 \times 2.5 \times 2 = 10$

Þ $v = \sqrt{10}$ m/s = **3.2 m/s** (to 1 decimal place.)

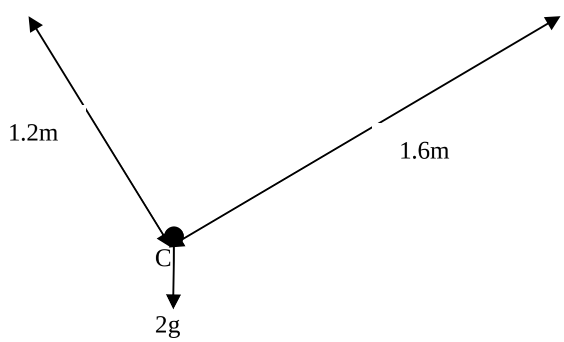
(v) $v^2 = u^2 + 2as$

Þ $0 = 10 - 2 \times 10 \times s$

Þ $20s = 10$

Þ $s = 0.5$ m.

17.



(i) $2.0^2 = 1.2^2 + 1.6^2$,

Þ **AB** is the **hypotenuse** of the triangle **ABC**

Þ angle **ACB** = **90º**.

(ii) From Δ **ABC**:

$$\mathbf{Cos\ \alpha = \frac{1.6}{2.0} = 0.8;}$$

$$\mathbf{Cos\ (90 - \alpha) = \frac{1.2}{2.0} = 0.6.}$$

Resolving **vertically**:

$$\mathbf{T_1 Cos\ \alpha + T_2 Cos\ (90 - \alpha) = 20}$$

Þ

$$\mathbf{0.8\ T_1 + 0.6\ T_2 = 20} \quad \ldots \quad (i)$$

Resolving **horizontally**:

$$\mathbf{T_1 Cos\ (90 - \alpha) = T_2 Cos\ \alpha}$$

Þ $$\mathbf{0.6\ T_1 = 0.8\ T_2}$$

Þ $$\mathbf{0.8\ T_2 - 0.6\ T_1 = 0} \quad \ldots \quad (ii)$$

Solving **simultaneously**:

(i) × 3 … $$\mathbf{2.4T_1 + 1.8\ T_2 = 60} \quad \ldots \quad (iii)$$

(ii) × 4 … $$\mathbf{-2.4T_1 + 3.2\ T_2 = 0} \quad \ldots \quad (iv)$$

(iii) + (iv) … $$\mathbf{5\ T_2 = 60}$$

Þ $$\mathbf{T_2 = 12N.}$$

Substitute in (ii): $$\mathbf{T_1 = \frac{0.8(12)}{0.6}}$$

Þ $$\mathbf{T_1 = 16N.}$$

(iii) $\mathbf{T_2 = 12N}$. (See part (ii) for working.)